

Laplace's Eqn. in Spherical Polar Co-ordinates

Laplace's Eqn. : $\nabla^2 V = 0$ in spherical polar co-ordinates reads :

$$1/r^2 \partial/\partial r (r^2 \partial V/\partial r) + 1/(r^2 \sin\theta) \partial/\partial \theta (\sin\theta \partial V/\partial \theta) + 1/(r^2 \sin^2\theta) \partial^2 V/\partial \phi^2 = 0$$

We shall restrict ourselves, to problems with axial (azimuthal) symmetry, i.e., assume V is ϕ -indep.

$$\Rightarrow 1/r^2 \partial/\partial r (r^2 \partial V/\partial r) + 1/(r^2 \sin\theta) \partial/\partial \theta (\sin\theta \partial V/\partial \theta) = 0$$

Assume as before, $V(r, \theta) = R(r) f(\theta)$

$$\Rightarrow f/r^2 d/dr (r^2 dR/dr) + R/(r^2 \sin\theta) d/d\theta (\sin\theta df/d\theta) = 0$$

Multiply both sides by r^2 and divide by $R(r) f(\theta)$:

$$\Rightarrow (1/R) d/dr (r^2 dR/dr) + (1/f \sin\theta) d/d\theta (\sin\theta df/d\theta) = 0$$

The 2nd term is purely a function of ' θ ' while 1st term is purely a function of ' r ', which is possible only if both are equal to some const., say ' λ '.

$$\Rightarrow (1/f \sin\theta) d/d\theta (\sin\theta df/d\theta) = \lambda \text{ ---- (1)}$$

$$\Rightarrow (1/R) d/dr (r^2 dR/dr) = -\lambda \text{ ---- (2)}$$

$$\text{Eqn.(1)} \Rightarrow (1/\sin\theta) d/d\theta (\sin\theta df/d\theta) = \lambda f$$

$$\text{Put } x = \cos\theta \Rightarrow df/d\theta = (df/dx) (dx/d\theta) = (df/dx) (-\sin\theta)$$

$$\Rightarrow (\sin\theta df/d\theta) = (df/dx)(-\sin^2\theta) = (df/dx)(x^2 - 1)$$

$$\Rightarrow d/d\theta (\sin\theta df/d\theta) = d/dx \{(x^2 - 1) df/dx\} dx/d\theta$$

$$= d/dx \{(x^2 - 1) df/dx\}(-\sin\theta)$$

$$\Rightarrow (1/\sin\theta) d/d\theta (\sin\theta df/d\theta) = d/dx \{(1 - x^2) df/dx\}$$

$$\text{Thus, eqn.(1)} \Rightarrow d/dx \{(1 - x^2) df/dx\} - \lambda f = 0,$$

$$\Rightarrow (1 - x^2) d^2f/dx^2 - 2x df/dx - \lambda f = 0, \text{ which is nothing but the Legendre eqn.}$$

We know, that if we wish to have a convergent solution for $x = \pm 1$, i.e., $\theta = 0$ and π , which we do wish, λ must be of the form : $-\ell(\ell + 1)$.

The corresponding solution is written as : $P_\ell(x) = P_\ell(\cos\theta)$

$$\text{Eqn.(2)} \Rightarrow d/dr (r^2 dR/dr) = -\lambda R = \ell(\ell + 1) R$$

$$\text{Try a solution of the form : } \mathbf{R} = \mathbf{r}^n \Rightarrow dR/dr = n \mathbf{r}^{n-1}$$

$$\Rightarrow r^2 dR/dr = n \mathbf{r}^{n+1}$$

$$\Rightarrow d/dr (r^2 dR/dr) = n(n + 1) \mathbf{r}^n = \ell(\ell + 1) \mathbf{r}^n$$

$$\Rightarrow n(n + 1) = \ell(\ell + 1)$$

$$\Rightarrow n^2 + n - \ell(\ell + 1) = 0$$

$$\Rightarrow n^2 + n(\ell + 1) - n\ell - \ell(\ell + 1) = 0$$

$$\Rightarrow \{n + (\ell + 1)\} \{n - \ell\} = 0 \Rightarrow \mathbf{n} = \ell, \text{ or, } \mathbf{n} = -(\ell + 1)$$

$$\text{i.e., } \mathbf{R} = \mathbf{A} \mathbf{r}^\ell + \mathbf{B} / \mathbf{r}^{\ell+1}$$

Thus, the general solution is : $\mathbf{V}(\mathbf{r}, \theta) = \Sigma (\mathbf{A}_\ell \mathbf{r}^\ell + \mathbf{B}_\ell / \mathbf{r}^{\ell+1}) \mathbf{P}_\ell(\cos\theta)$

Example : A grounded conducting sphere, placed in a uniform electric field

$$\text{For a uniform electric field } \mathbf{E} = E_0 \mathbf{k} \Rightarrow V = -E_0 z = -E_0 r \cos\theta + C$$

i) For $r \rightarrow \infty$, $V = -E_0 r \cos\theta + C$

ii) For $r = a$, $V = 0$ (since the conducting sphere is earthed)

For $r \rightarrow \infty$, $V(r, \theta) = \sum (A_\ell r^\ell) P_\ell(\cos\theta) = A_0 + A_1 r \cos\theta + \dots = -E_0 r \cos\theta + C$

$\Rightarrow A_0 = C$, $A_1 = -E_0$, and $A_\ell = 0$ for all other ℓ .

$\Rightarrow V(r, \theta) = C - E_0 r \cos\theta + \sum B_\ell / r^{\ell+1} P_\ell(\cos\theta)$

At $r = a$: $V(r, \theta) = C - E_0 a \cos\theta + \sum B_\ell / a^{\ell+1} P_\ell(\cos\theta)$

$\Rightarrow 0 = C P_0(\cos\theta) - E_0 a P_1(\cos\theta) + \sum B_\ell / a^{\ell+1} P_\ell(\cos\theta)$

We can compare term by term, because Legendre polynomials are all linearly independent.

$\Rightarrow C = -B_0/a$, $E_0 a = B_1/a^2$ and $B_\ell = 0$ for all other ' ℓ '

$\Rightarrow B_0 = -C a$, $B_1 = E_0 a^3$

$\Rightarrow V(r, \theta) = C - E_0 r \cos\theta - C a/r + E_0 a^3/r^2 \cos\theta$

$= C(1 - a/r) + E_0 (-r + a^3/r^2) \cos\theta.$