Wave Motion: Acoustic plane waves

Wave:

A wave is a disturbance moving through the medium. The nature if disturbance is different for different kind of waves. In material medium, the disturbance may relate to displacement, velocity and acceleration of its particle, to the variation of density, pressure or any other property of the medium, for example its temperature. In a wave motion, it is the disturbance (disturbed condition) of the medium which moves; but the medium itself is not bodily transferred.

There may be waves without medium, such as electromagnetic waves. Radio waves, light, heat radiation, X-rays etc. are the examples of this kind of waves. But we shall confine ourselves to elastic waves in material.

The medium transmitting an elastic wave must have inertia and elasticity distributed through it. The waves originate in displacement of some portion of an elastic medium from its equilibrium position. The elasticity of the medium gives rise to restoring forces which tend to restore a displaced element to its equilibrium position. But the inertia of the element carries it beyond, causes the element to oscillate about its equilibrium position, until viscous forces bring it to rest. A displaced element also displaces a neighbouring element due to elasticity. In this way the disturbance is transmitted from one layer to the next layer, and progresses through the medium.

The disturbance at any point of a medium through which a wave is passing is a function of time. At a given time, the disturbances at different points are function of position. Therefore, a wave is a time-varying quantity which is also function of position. A mathematical expression is said to describe a wave motion if it gives the state of disturbance for all points at all instant of time.

Since every disturbance in a medium involves energy, wave propagation implies transfer of energy.

Difference between vibration and wave motion:

Like wave motion, vibration requires inertia and elasticity; but both are localized in the vibrating system. Energy is also localised in vibrating system. When a mass attached to a spring vibrates, there is inertia in the mass and elasticity in the spring. The system vibrates and energy resides in it. In wave motion, inertia and elasticity are distributed over the medium and energy is transferred from the affected parts of the medium to the unaffected parts.

Example: The common example of wave is when we throw a stone on still water surface. Ripples are generated on water surface. These ripples spread out on the surface on ever-expanding circles. Here the disturbance is the displacement of water particles on the surface. The restoring force is provided by surface tension and gravity.

Waves of different kinds:

Waves may be classified according to the direction of disturbance relative to the direction of propagation of the wave. Waves in which the direction of the disturbance is parallel to the line of propagation are called longitudinal waves. Sound waves in solids, liquids and gases, compressional waves along a spring etc. are examples of longitudinal waves. Waves in which the disturbance is in a plane perpendicular to the line of propagation are called transverse wave. Waves along a stretched string, electromagnetic waves etc. are examples of transverse waves.

Characteristics of a periodic wave of constant type:

A graph showing the displacement of a particle from its mean position as a function of time is called time-displacement curve. In a wave of constant type and without attenuation, all particles in the path of propagation will have the same time-displacement curve. But a particle away from the source of disturbance will acquire at some later instant the motion of a particle preceding it. In a homogeneous medium, this lag is proportional to the distance between the particles.

For particles lying on the line of propagation, we may draw a graph for some particular instant, showing the displacement (y) of the particle as a function of position (x) (i.e. their distance from the source or from some arbitrary point on the line of propagation). Such a curve is called a space-displacement curve, or wave-form or wave-profile. A wave of constant type is one in which the waveform does not change. The velocity with which an unchanging wave-form moves is called the wave velocity or the phase velocity.

Let a particle at position x_2 on the line of propagation acquires at some later instant t_2 the same state of motion which another particle at position x_1 near to the source had at some earlier instant t_1. The phase of the motion has moved from x_1 to x_2 in time $(t_2 - t_1)$. The phase velocity is $(x_2 - x_1)/(t_2 - t_1)$. Particles of the medium merely oscillate about their mean position.

From a study of time- and space- displacement curves of a periodic wave of constant we find that

(a) The motion of a given particle is repeated at equal interval of time. This time is called the time period or periodic time (T). The number of complete oscillations executed by the particle in one second is called the frequency. It is given by

$$n = \frac{1}{T}$$

(b) Particles separated by a certain distance, or any integral multiple of it, are in the same state of motion i.e. in the same phase. The shortest distance between two particles in the same phase along the line of propagation is called the wavelength (λ).

A periodic wave is thus characterised by two periodicities, one in time and the other in space. The time period (T) gives the periodicity in time and wavelength (λ) gives the periodicity in space.

Relation between T and λ:

Let us consider a particle which is just disturbed by a progressive wave of velocity c. By the time it executes a complete oscillation (in time T) the wave advances through a distance cT. At the end of this period the particle acquires the same phase as it had at the beginning of the period. This is also the phase of the particle which the wave has just reached at. According to the definition of the wavelength the distance between the two particles (nearest particles in the same phase) is one wavelength. But this distance is also cT.

$$\therefore \lambda = cT = \frac{c}{n}$$

$$\therefore c = n\lambda = \frac{\lambda}{T}$$

n is called the frequency of the wave. It is the number of waves emitted by the source in one second, or number of waves in distance c.

Wave-front:

A surface at all points of which the phase of the particle is the same at any given instant is called a wave-front. We may consider the propagation of a wave through a medium as the motion of wave-fronts with velocity equal to the wave velocity. In case of a point source or pulsating sphere in an isotropic medium, the wave-fronts are spherical surfaces. They are called the spherical wave-fronts. For linear sources the wave-fronts are cylindrical if the end effects are ignored. When the loci of points of constant phase lie on parallel plane surfaces, the wave-front is called plane. A plane wave may be defined as one that moves in fixed direction without spreading.

A ray is the path of an element o f the wave-front. In an isotropic medium rays are perpendicular to the wave-fronts. For normal waves rays are parallel lines normal to the wave-front. For spherical waves, they are the radii.

Characteristics of a plane progressive wave of simple harmonic type:

(i) Every particle describes simple harmonic motion along the line of propagation of the wave; there is change of phase from point to point.

(ii) The arrangement without changing its type advances with a uniform velocity, its value depending on the elastic constant and the density of the medium.

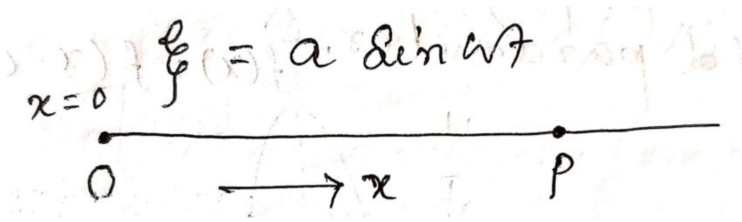
(iii) Any particular displacement any particular instant is repeated at regular distances called the wavelength. The velocity and acceleration of the particles of a wavelength apart from one another are the same.

(iv) If n is the number of vibrations per second (frequency) of a particle along the direction of propagation of the wave, then $n\lambda = c$, where c is the velocity of the wave and λ is the wavelength.

Equation of a plane progressive wave:

Let us consider a wave propagating along the positive direction of X-axis with velocity c. Let displacement at any instant t at x=0 be

$\xi = aSin\omega t$ ....(1)



In time t the wave has travelled from point O to point P without changing its form. We have to find the displacement at P at same instant t. The wave takes time $\frac{x}{c}$ to move from O to P. Hence displacement at O at time ($t-\frac{x}{c}$) is same as that at P at time t. Thus displacement at P at any instant t is given by

$$\xi = aSin\omega(t-\frac{x}{c}) \text{ ....(2)}$$

If the wave moves towards the negative direction of X-axis, the displacement at P at any instant t will be the same as that at O at ($t+\frac{x}{c}$). Hence displacement at P in this case is given by

$$\xi = aSin\omega(t+\frac{x}{c}) \text{ ....(3)}$$

Now, $\omega = \frac{2\pi}{T} = 2\pi n = \frac{2\pi c}{\lambda}$ $[\because c = n\lambda]$

Hence equation (2) can be written as

$$\xi = aSin\frac{2\pi c}{\lambda}(t-\frac{x}{c}) = aSin\frac{2\pi}{\lambda}(ct-x)$$

Therefore, equation of a wave moving along +ve direction of X-axis is given by

$$\xi = aSin\frac{2\pi}{\lambda}(ct-x)$$

And along –ve direction of X-axis is given by

$$\xi = aSin\frac{2\pi}{\lambda}(ct+x)$$

These equations can also be written in the following form

$$\xi = aSin(\omega t-kx)$$

$$\xi = aSin(\omega t+kx)$$

Where $\mathrm{k} = \dfrac{2\pi}{\lambda}$, and is called the wave number.

The differential wave equation for plane waves of constant wave-form in one dimension:

Let $\psi = f(ct - x)$ be any wave-field parameter.

Writing $z = ct - x$

$$f'(z) = \frac{d}{dz} f(z) = \frac{d\psi}{dz}$$

And $f''(z) = \dfrac{d}{dz} f'(z) = \dfrac{d^2\psi}{dz^2}$

Now, $z = ct - x$

$$\therefore \frac{\partial z}{\partial x} = -1, \frac{\partial z}{\partial t} = c$$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{d\psi}{dz}\frac{\partial z}{\partial x} = -\frac{d\psi}{dz} = -f'(z)$$

And $\dfrac{\partial^2\psi}{\partial x^2} = \dfrac{\partial}{\partial x}\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial}{\partial x} f'(z) = -\dfrac{d}{dz} f'(z)\dfrac{\partial z}{\partial x} = f''(z)$ ....(1)

Again, $\dfrac{\partial \psi}{\partial t} = \dfrac{d\psi}{dz}\dfrac{\partial z}{\partial t} = c\dfrac{d\psi}{dz} = cf'(z)$

And $\dfrac{\partial^2\psi}{\partial t^2} = \dfrac{\partial}{\partial t}\dfrac{\partial \psi}{\partial t} = \dfrac{\partial}{\partial t} cf'(z) = c\dfrac{d}{dz} f'(z)\dfrac{\partial z}{\partial t} = c^2 f''(z)$ ....(2)

Comparing equations (1) and (2) we can write

$\dfrac{\partial^2\psi}{\partial t^2} = c^2 \dfrac{\partial^2\psi}{\partial x^2}$ ....(3)

Or, $\dfrac{\partial^2\psi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\psi}{\partial t^2}$ ....(4)

Here c is the wave velocity.

In exactly similar way we can show that any function of ($ct + x$) leads to the same differential equation. Thus any function of ($ct \pm x$) will satisfy the above differential equation.

The above equation is the differential wave equation for plane waves of constant type. Its general solution is $f_1(ct - x) + f_2(ct + x)$, where f_1 and f_2 are arbitrary functions. The general solution represents two plane waves moving in opposite directions with velocity c. The waves need not have the same wave-form since theoretically f_1 and f_2 may be different functions. In practice, however, the wave-form and the functions f_1 and f_2 are similar.

It is the simplest differential wave equation in one dimension, and has its limitations. For example, it cannot represent waves of large amplitude, flexural waves, attenuated waves etc.

Note: Solution of the differential wave equation in one dimension can have solution in the following forms

$$\psi(x,\mathrm{t}) = aSin\frac{2\pi}{\lambda}(ct \mp x) = aSink(ct \mp x) = aSin(\omega t \mp kx)$$
$$\psi(x,\mathrm{t}) = aCos\frac{2\pi}{\lambda}(ct \mp x) = aCosk(ct \mp x) = aCos(\omega t \mp kx)$$
$$\psi(x,\mathrm{t}) = ae^{ik(ct \mp x)} = ae^{i(\omega t \mp kx)}$$

Particle velocity:

Displacement of the particle which is located at x is given by

$$\xi = aSin(\omega t - kx)$$

Hence, the velocity of the particle is given by

$$\mathrm{u} = \frac{\partial \xi}{\partial t} = a\omega Cos(\omega t - kx)$$

Some definitions:

When a wave of compression and rarefaction passes through a medium, volume, density and fluctuate locally about the normal value.

(i) Dilatation (Δ):

It is the ratio of the increment of volume (δV) to the original volume (V_0). If V is the instantaneous volume then

$\Delta = \dfrac{\delta V}{V_0}$ and $V = V_0 + \delta V = V_0(1+\Delta)$.

Δ is thus volume strain.

(ii) Condensation (s):

It is the ratio of the increment in density ($\delta\rho$) to the original density (ρ_0). If ρ be the instantaneous density, then

$s = \frac{\delta\rho}{\rho_0}$ and $\rho = \rho_0 + \delta\rho = \rho_0(1+s)$.

Since we generally consider a thin slab or layer of fixed mass,

$$\rho V = \rho_0(1+s).V_0(1+\Delta) = \rho_0 V_0$$

$$\therefore (1+s)(1+\Delta) = 1$$

As s and Δ are very small fractions of first order, $s\Delta$ is a small fraction of the second order and hence can be neglected.

$$\therefore 1+s+\Delta+s\Delta = 1$$

$$\therefore s = -\Delta$$

(iii) Excess pressure (p):

When a layer of fluid medium is compressed during the passage of a compressional wave, the pressure in the layer rises from the normal value P_0 to some value P . $p = \delta p = P - P_0$ is called the excess pressure. In a rarefaction P is smaller than P_0 and p is negative.

In the case of sound wave, p is very important quantity as most of the measurements in sound are made with its help. We call it as sound pressure or acoustic pressure.

(iv) Bulk modulus (K):

If a change of pressure increases a volume V_0 by δV , the strain is $\frac{\delta V}{V_0}$, and stress = change of pressure = $-\delta p$, bulk modulus K is defined as the ratio of this stress to the strain. For infinitesimal changes,

$$K = -\frac{\delta p}{\frac{\delta V}{V_0}} = -V_0\frac{\delta p}{\delta V}$$

In a sound field, $\delta p = p$, the excess pressure and $\frac{\delta V}{V_0} = \Delta = -s$, when s is very small.

$$\therefore K = -\frac{p}{\Delta} = \frac{p}{s}$$

$$\therefore p = Ks$$

Note: In a plane wave, the quantities that vary with x and t, and obey the differential wave equation are p , ξ , u , $\delta\rho$, s and Δ .

An important property of the wave equation

$$\frac{\partial^2\psi}{\partial t^2} = c^2 \frac{\partial^2\psi}{\partial x^2}$$

Is that $\dot{\psi} = \frac{\partial\psi}{\partial t}$ and $\ddot{\psi} = \frac{\partial^2\psi}{\partial t^2}$ etc. satisfy the same differential wave equation as ψ satisfies. This is also true for the quantities $\frac{\partial\psi}{\partial x}$, $\frac{\partial^2\psi}{\partial x^2}$ etc. All of them propagate with the same velocity.

As example, if we differentiate the wave equation with respect to t, we have

$$\frac{\partial}{\partial t}.\frac{\partial^2\psi}{\partial t^2} = c^2 \frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial x^2}$$

Or, $\frac{\partial^2}{\partial t^2}.(\frac{\partial\psi}{\partial t}) = c^2 \frac{\partial^2}{\partial x^2}.(\frac{\partial\psi}{\partial t})$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Thus, we see that particle velocity satisfies the same differential wave equation. Similarly, we can show that above mentioned quantities also satisfy the same differential wave equation.

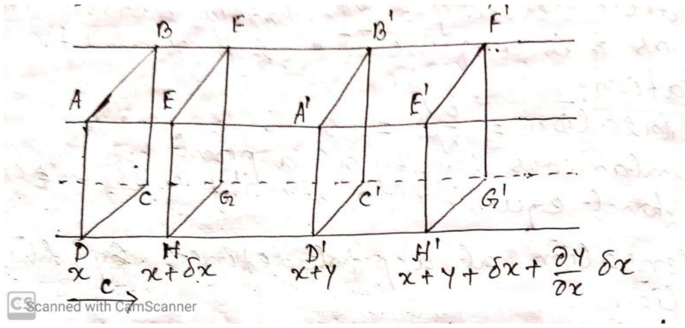
Propagation of a longitudinal plane progressive wave in a fluid medium:

The following simplifying assumptions are made for deriving an expression for the velocity of plane longitudinal waves in a fluid.

(i) The medium is homogeneous, isotropic, and has no dissipative forces which may arise from viscosity and thermal conduction.

(ii) In the equilibrium condition the pressure P_0 and density ρ_0 are same everywhere (i.e. we neglect the effect of gravity).

(iii) The strain developed in the medium by the wave is so small that Hooke's law is obeyed i.e. Δ and s <<1.



Let ABCD represents a cross-section of an elastic fluid medium having cross-sectional area α perpendicular to the direction of propagation i.e. along the X-axis. EFGH is another layer parallel to the previous layer at a distance δx (δx <<λ) from the previous one. Let the positions of the layers are x and $(x+\delta x)$ from a fixed point to the left. Since the wave is plane progressive, when the wave reaches the plane ABCD, displaces the particles at that plane by some distance say y i.e. the plane takes the new position A'B'C'D'. Similarly, the plane EFGH is displaced to E'F'G'H', then the displacement being $(y+\frac{\partial y}{\partial x}\delta x)$.

So the change in thickness of the layer is $[(x+y+\delta x+\frac{\partial y}{\partial x}\delta x)-(x+y)]-[(\mathrm{x}+\delta x)-x]=\frac{\partial y}{\partial x}\delta x$

and the increase in volume of the layer is $\delta V=\alpha\frac{\partial y}{\partial x}\delta x$.

The original volume, $V=\alpha\delta x$.

Hence, the volume strain = $\frac{\delta V}{V}=\frac{\alpha\frac{\partial y}{\partial x}\delta x}{\alpha\delta x}=\frac{\partial y}{\partial x}$.

If the volume stress be p (excess pressure), then the bulk modulus

$$K=\frac{-p}{\frac{\partial y}{\partial x}}$$

$$\therefore\ p=-K\frac{\partial y}{\partial x}$$

The negative sign is due to the fact that an increase in pressure results in decrease in in volume and vice-versa.

Hence the excess pressure on the layer ABCD is $-K\frac{\partial y}{\partial x}$.

Excess pressure on the layer EFGH is $p+\frac{\partial p}{\partial x}\delta x=-K\frac{\partial y}{\partial x}+\frac{\partial}{\partial x}(-K\frac{\partial y}{\partial x})\delta x=-K\frac{\partial y}{\partial x}-K\frac{\partial^2 y}{\partial x^2}\delta x$.

Hence the difference of excess pressure is

$$-K\frac{\partial y}{\partial x}-(-K\frac{\partial y}{\partial x}-K\frac{\partial^2 y}{\partial x^2}\delta x)=K\frac{\partial^2 y}{\partial x^2}\delta x$$

The net force in the positive direction of the X-axis is

$$=K\alpha\frac{\partial^2 y}{\partial x^2}\delta x\,.$$

If ρ be the density of the fluid, then by Newton's law

$$K\alpha\frac{\partial^2 y}{\partial x^2}\delta x=(\rho\alpha\delta x).\frac{\partial^2 y}{\partial t^2}$$

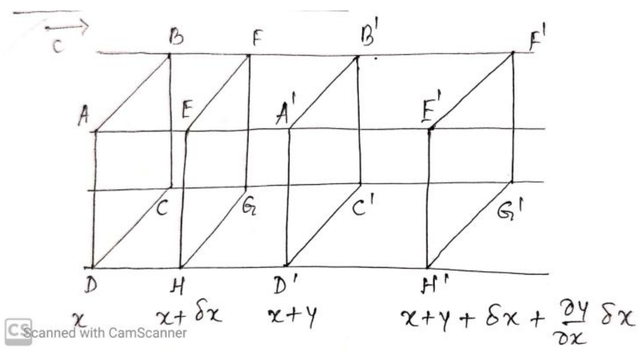
Or, $K\frac{\partial^2 y}{\partial x^2}=\rho\frac{\partial^2 y}{\partial t^2}$

$$\therefore\frac{\partial^2 y}{\partial t^2}=\frac{K}{\rho}\frac{\partial^2 y}{\partial x^2}$$

$\therefore\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$, where $c=\sqrt{\frac{K}{\rho}}$ is the wave velocity.

This is the required differential wave equation.

Plane progressive longitudinal wave propagating through a solid bar:



We consider a very long solid bar of material of Young's modulus Y . Let α be the cross-sectional area of the bar. Now, we consider two transverse planes ABCD and EFGH in the direction of propagation of the wave at positions x and $x+\delta x$ respectively with respect to some fixed plane to the left. So the thickness of the layer formed by two planes is δx . When the longitudinal plane progressive wave propagates through the bar along X-direction, plane ABCD is displaced by y at any instant t to a new position A'B'C'D'. Similarly, the plane EFGH is displaced by $(y+\frac{\partial y}{\partial x}\delta x)$ to a new position E'F'G'H'.

Now, change in thickness of the layer = $[(x+y+\delta x+\frac{\partial y}{\partial x}\delta x)-(x+y)]-[(x+\delta x)-x]=\frac{\partial y}{\partial x}\delta x$

$\therefore$ Longitudinal strain = $\frac{\frac{\partial y}{\partial x}\delta x}{\delta x}=\frac{\partial y}{\partial x}$

Therefore, longitudinal stress = $Y\frac{\partial y}{\partial x}$, which produces the acoustic pressure of the bar. So, the acoustic force acting on the layer ABCD due to the material of the bar lying on the left hand side of ABCD is $Y\alpha\frac{\partial y}{\partial x}$. This force is along the negative direction of X-axis. The force acting on the layer EFGH by portion of the bar lying on the right hand side of EFGH along positive direction of X-axis is $Y\alpha\frac{\partial y}{\partial x}+\frac{\partial}{\partial x}(Y\alpha\frac{\partial y}{\partial x})\delta x$.

Hence the net force acting on the element of the bar in the positive direction of X-axis is

$$=[Y\alpha\frac{\partial y}{\partial x}+\frac{\partial}{\partial x}(Y\alpha\frac{\partial y}{\partial x})\delta x]-Y\alpha\frac{\partial y}{\partial x}=Y\alpha\frac{\partial^2 y}{\partial x^2}\delta x.$$

According to Newton's second law, this must be equal to mass x acceleration.

$$\therefore Y\alpha\frac{\partial^2 y}{\partial x^2}\delta x=(\rho\alpha\delta x).\frac{\partial^2 y}{\partial t^2}$$

Or, $Y\frac{\partial^2 y}{\partial x^2}=\rho\frac{\partial^2 y}{\partial t^2}$

$$\therefore \frac{\partial^2 y}{\partial t^2}=\frac{Y}{\rho}\frac{\partial^2 y}{\partial x^2}$$

$\therefore \frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$, where $c=\sqrt{\frac{Y}{\rho}}$ is the velocity of the wave.

This is the required differential wave equation.

Spherical Wave:

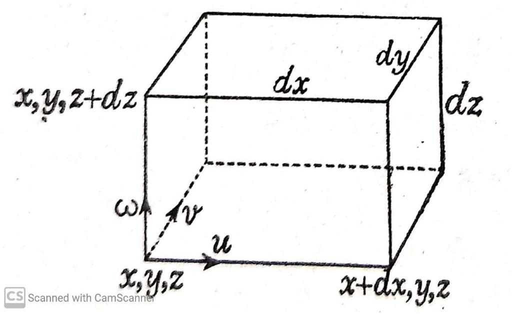
A large surface vibrating perpendicular to itself generates practically plane waves. A sound beam at a large distance from a source also provides practically plane waves.

Plane wave is one which does not diverge laterally. Unfortunately most of the sound sources produce divergent waves. For small sources, waves are practically spherical. As the same energy is spread out over a large surface with increasing distance from the source, both the sound pressure and intensity falls off rapidly.

Now, we want to derive the wave equation for spherical waves. We may extend the one-dimensional wave equation to three dimensions and this can be easily done in Cartesian co-ordinates. The co-ordinates then may be exchanged to spherical polar co-ordinates for spherical waves and to cylindrical-polar co-ordinates for cylindrical waves.

All three basic wave equations make use of three basic relations- (i) equation of continuity, (ii) relation expressing the elastic constant to the medium, (iii) Newton's second law of motion. In all cases changes in the medium due to the passing of the wave is assumed to be small.

Equation of continuity:



Les us consider a volume element $dxdydz$ at the location (x,y,z) in a medium which is in some state of motion. Now, the motion of the medium does not destroy any part of its mass. Therefore, the rate of flow of mass out of the volume element must be equal to the rate of decrease of mass within the volume. This is the physical content of equation of continuity.

Let u , v and w be the components of the velocity of the fluid medium at the point (x,y,z). The need not be constants, and taken as functions of x,y,z respectively.

We consider the face of the parallelepiped perpendicular to the X-axis at x. Let ρ be the density of fluid at the face. At the parallel face at $x+dx$, both ρ and u are taken as different. Then the mass flowing into the volume element through the first face per unit time is $(\rho u)dydz$. Mass flowing out through the opposite face per unit time is $[\rho u+\frac{\partial}{\partial x}(\rho u)dx]dydz$. Hence, the net rate of flow of fluid mass through the pair of opposite faces is

$$[\rho u+\frac{\partial}{\partial x}(\rho u)dx]dydz-(\rho u)dydz=\frac{\partial}{\partial x}(\rho u)dxdydz .$$

In similar way, we find that the net rate of flow of mass per unit time through pair of faces perpendicular to Y-axis and Z-axis are $\frac{\partial}{\partial y}(\rho v)dydxdz$ and $\frac{\partial}{\partial z}(\rho w)dzdxdy$ respectively. The sum of these three is the net rate of flow of fluid mass per unit time out of the volume element. The rate of change density is $\frac{\partial \rho}{\partial t}$, and the total loss of mass to the volume element per unit time is $-\frac{\partial \rho}{\partial t}dxdydz$. Equating these rates of flow of fluid mass, we get

$$[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)]dxdydz=-\frac{\partial \rho}{\partial t}dxdydz$$

$$\therefore \frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=-\frac{\partial \rho}{\partial t} \text{ ....(1)}$$

$$\therefore \vec{\nabla}.(\rho\vec{v})=-\frac{\partial \rho}{\partial t} \text{ ....(2)}$$

This is the equation of continuity and is nothing but the statement of conservation of mass.

Simpler form of equation of continuity:

Eq. (1) can be simplified by using certain valid assumptions. If we assume that the change in density ρ of fluid from point to point is so small that it can be neglected, and we can write

$$\frac{\partial}{\partial x}(\rho u)\approx \rho\frac{\partial u}{\partial x} \text{ ....(3)}$$

Again, since $\rho=\rho_0(1+s)$, and s is small quantity, eq. (3) can be further simplified as

$$\frac{\partial}{\partial x}(\rho u)\approx \rho_0\frac{\partial u}{\partial x}$$

Hence, by using the simplifications eq. (1) is reduced to

$$\rho_0(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=-\frac{\partial \rho}{\partial t}$$

Using the relation $\rho=\rho_0(1+s)$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{\partial s}{\partial t} \text{ ....(4)}$$

Application of Newton's second law of motion:

When a longitudinal wave passes through a medium, there is a variation of pressure from point to point. Let the excess pressure at (x,y,z) be p. The excess pressure at $x+dx$ is $p+\frac{\partial p}{\partial x}dx$. The force on the face at x is $pdydz$. That on the opposite face at $x+dx$ is $-(p+\frac{\partial p}{\partial x}dx)dydz$. These two forces produce an unbalanced force $pdydz-(p+\frac{\partial p}{\partial x}dx)dydz=-\frac{\partial p}{\partial x}dxdydz$ on the volume element.

By Newton's second law, the rate of change of momentum is equal to the applied force. The X-component of the momentum of the volume element is $\rho udxdydz$. Hence,

$$-\frac{\partial p}{\partial x}dxdydz=\frac{\partial}{\partial t}(\rho u)dxdydz$$

$$\therefore -\frac{\partial p}{\partial x}=\frac{\partial}{\partial t}(\rho u) \text{........................(5)}$$

In similar way, we can show that

$$-\frac{\partial p}{\partial y}=\frac{\partial}{\partial t}(\rho v) \text{.................................(6)}$$

And

$$-\frac{\partial p}{\partial z}=\frac{\partial}{\partial t}(\rho w) \text{..(7)}$$

Differential equation of the wave in three dimensions:

Differentiating eqns. (5), (6) and (7) with respect to x, y and z respectively and adding, we get

$$-(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2})=\frac{\partial}{\partial x}.\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial y}.\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial z}.\frac{\partial}{\partial t}(\rho w) \text{.........................(8)}$$

Differentiating eq. (1) wrt t, we get

$$\frac{\partial}{\partial t}.\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial t}.\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial t}.\frac{\partial}{\partial z}(\rho w)=-\frac{\partial^2 \rho}{\partial t^2} \text{........................(9)}$$

Since the operations of partial differentiations are commutative, by comparing eqns. (8) and (9), we get

$$\frac{\partial^2 \rho}{\partial t^2}=\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2} \text{...................................(10)}$$

We now take the help of elastic property of the medium. For small value of the condensation s , the excess pressure p and the instantaneous density ρ are related by the relations

$p = Ks$ and $\rho = \rho_0(1+s)$, where K is the bulk modulus and ρ_0 is the equilibrium density of the medium.

$$\therefore \rho = \rho_0(1+\frac{p}{K})$$

Differentiating ρ twice wrt t, we get

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\rho_0}{K}\frac{\partial^2 p}{\partial t^2}$$

$$\therefore \frac{\partial^2 \rho}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \text{.....................................(11)}$$

where $c = \sqrt{\frac{K}{\rho_0}}$ is the wave velocity.

Now, substituting the value of $\frac{\partial^2 \rho}{\partial t^2}$ from eq. (11) into eq. (10), we get

$$\frac{\partial^2 p}{\partial t^2} = c^2(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}) = c^2\nabla^2 p \text{ ....(12)}$$

This is the differential acoustic wave equation in three dimensions in terms of acoustic pressure/excess pressure.

In spherical polar co-ordinates, Laplacian operator takes the form

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial \theta}(Sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

When there is spherical symmetry, the Laplacian operator takes the form

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}$$

Hence, in the case of spherical waves, the wave equation (12) reduces to the form

$$\frac{\partial^2 p}{\partial t^2} = c^2(\frac{\partial^2 p}{\partial r^2}+\frac{2}{r}\frac{\partial p}{\partial r}) = \frac{c^2}{r}\frac{\partial^2 (rp)}{\partial r^2}$$

Since r and t are independent variables, we may write the above equation as

$$\frac{\partial^2 (rp)}{\partial t^2} = c^2 \frac{\partial^2 (rp)}{\partial r^2} \text{..............................(13)}$$

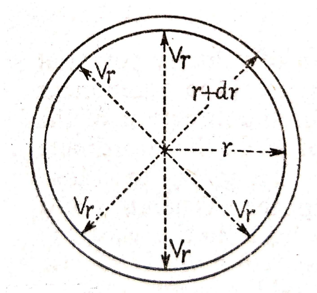
This is the spherical wave equation in terms of acoustic pressure.

Note: It is easier to derive a wave equation when the wave-field parameter is a scalar quantity than when it is a vector. Deriving the wave equations in terms of particle displacement $\vec{D}$ with components ξ, η, ζ or the particle velocity $\vec{V}$ with components u, v, w is much more difficult. The particle displacement ξ or particle velocity u in a plane wave is also a vector. But as they have fixed direction in a compressional wave they practically behave as scalars. So the deduction was easy.

Alternative way:

In the earlier method we used Cartesian co-ordinates to derive the general wave equation in three dimensions. We then used the form of the Laplacian operator which represents the spherical symmetry in order to get the spherical wave equation. Instead of doing so, we can assume spherical symmetry from the beginning, and express the equation of continuity and the dynamic equation in forms appropriate to such symmetry. These, combined with the elastic relation $p = Ks$ and $\rho = \rho_0(1+s)$ will lead us to the desired spherical wave equation.

Equation of continuity for spherical waves:



In purely spherical waves the flow is radial. For a given time t, the field parameters are functions of r only. Let us consider a thin spherical shell of radii r and $r + dr$ as volume element. If ρ be the instantaneous density, the rate of flow of mass into it is $4\pi r^2 \rho V_r$, where V_r is the radial flow velocity at distance r from the centre of the sphere. At the outer surface ρV_r has the value $\rho V_r + \frac{\partial}{\partial r}(\rho V_r)dr$ .Hence, the rate of flow of mass out of the shell is

$$4\pi(r+dr)^2\{\rho V_r + \frac{\partial}{\partial r}(\rho V_r)dr\} \approx 4\pi\{r^2 \rho V_r + r^2 \frac{\partial}{\partial r}(\rho V_r)dr + 2r\rho V_r dr\}$$

[Since dr is very small, we have neglected the terms containing second order in smallness and retained terms containing only first order in smallness]

Hence, the net rate of flow of mass out of the element is

$$4\pi\{r^2\rho V_r + r^2\frac{\partial}{\partial r}(\rho V_r)dr + 2r\rho V_r dr\} - 4\pi r^2\rho V_r = 4\pi\{r^2\frac{\partial}{\partial r}(\rho V_r)dr + 2r\rho V_r dr\}.$$

This must be equal to the rate of loss of mass in the element i.e. $-4\pi r^2 dr\frac{\partial\rho}{\partial t}$.

$$\therefore 4\pi\{r^2\frac{\partial}{\partial r}(\rho V_r)dr + 2r\rho V_r dr\} = -4\pi r^2 dr\frac{\partial\rho}{\partial t}$$

Or, $\frac{\partial}{\partial r}(\rho V_r) + \frac{2}{r}(\rho V_r) = -\frac{\partial\rho}{\partial t}$ ....(1)

This is the equation of continuity for spherical waves.

For small amplitude waves we may replace density ρ of the medium in the left side of eq. (1) by its equilibrium value ρ_0 (here we are ignoring quantities of second order in smallness compared to first order in smallness). Since $\rho = \rho_0(1+s)$, we have $\frac{\partial\rho}{\partial t} = \rho_0\frac{\partial s}{\partial t}$. Hence we can write eq. (1) as

$\frac{\partial V_r}{\partial r} + \frac{2}{r}V_r = -\frac{\partial s}{\partial t}$ ....(2)

The dynamical equation:

If P is the instantaneous pressure at the inner face of the shell, that at the outer face of the shell may be written as $P + \frac{\partial P}{\partial r}dr$. Since $P = P_0 + p$ where P_0 is the equilibrium pressure and p is the excess pressure or acoustic pressure of the medium, $\frac{\partial P}{\partial r} = \frac{\partial p}{\partial r}$. The unbalanced force which is acting on the shell in the inward direction is $[(P + \frac{\partial P}{\partial r}dr) - P].4\pi r^2 = \frac{\partial P}{\partial r}dr.4\pi r^2 = 4\pi r^2\frac{\partial p}{\partial r}dr$. This must be equal to the product mass x acceleration of the shell. Hence

$$-4\pi r^2\frac{\partial p}{\partial r}dr = (4\pi r^2 dr\rho).\frac{\partial V_r}{\partial t}$$

Or, $\frac{\partial p}{\partial r} = -\rho\frac{\partial V_r}{\partial t} = -\rho_0\frac{\partial V_r}{\partial t}$ ....(3)

This is the relevant dynamic equation.

The elastic relation:

$p = Ks$ and $\rho = \rho_0(1+s)$, where K is the bulk modulus and ρ_0 is the equilibrium density of the medium.

The spherical wave equation in terms of p :

Now, differentiating the modified equation of continuity with respect to t, we get

$$\frac{\partial^2 s}{\partial t^2} = -\frac{\partial}{\partial t}.\frac{\partial V_r}{\partial r} - \frac{2}{r}\frac{\partial V_r}{\partial t}$$

Or, $\frac{\partial^2 s}{\partial t^2} = -\frac{\partial}{\partial r}.\frac{\partial V_r}{\partial t} - \frac{2}{r}\frac{\partial V_r}{\partial t}$ ....(4)

[Since the order of partial differentiations commute]

Now, substituting for $\frac{\partial V_r}{\partial t}$ from eq. (3) in eq. (4) we get

$$\frac{\partial^2 s}{\partial t^2} = \frac{1}{\rho_0}(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}) \text{ ....(5)}$$

From $p = Ks$, we have $\frac{\partial^2 p}{\partial t^2} = K\frac{\partial^2 s}{\partial t^2}$.

$$\therefore \frac{\partial^2 p}{\partial t^2} = \frac{K}{\rho_0}(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r})$$

Or, $\frac{\partial^2 p}{\partial t^2} = c^2(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r})$ [Here $c = \sqrt{\frac{K}{\rho_0}}$ is the wave velocity]

Or, $\frac{\partial^2 p}{\partial t^2} = c^2\frac{1}{r}\frac{\partial^2}{\partial r^2}(rp)$

Since r and t are independent variables, we may write the above equation as

$$\frac{\partial^2 (rp)}{\partial t^2} = c^2\frac{\partial^2 (rp)}{\partial r^2} \text{ ....(6)}$$

This is the spherical wave equation in terms of acoustic pressure/excess pressure.

Solution of the spherical wave equation:

The spherical wave equation in terms of acoustic pressure is given by

$$\frac{\partial^2 (rp)}{\partial t^2} = c^2\frac{\partial^2 (rp)}{\partial r^2} \text{ ....(1)}$$

If we compare this equation with the plane wave equation

$$\frac{\partial^2\psi}{\partial t^2}=c^2\frac{\partial^2\psi}{\partial x^2}$$

we notice that they have the same form except rp replaces ψ in the numerator and r replaces x in the denominator. Solution of the spherical wave equation can be written as

$$rp=f(ct\mp r)$$

$$\therefore p=\frac{1}{r}f(ct\mp r)$$

Where f is an arbitrary function of $(ct-r)$ or $(ct+r)$. The general solution may be written as

$$p=\frac{A}{r}f(ct-r)+\frac{B}{r}f(ct+r)\text{................(2)}$$

where A and B are two arbitrary constants. The first term on the right hand side represents a wave moving along r with velocity c. It is a divergent spherical wave. The second term represents a wave moving towards the origin, and hence is a convergent spherical wave. The later is not of much practical importance.

The solution of the divergent wave shows that the acoustic pressure decreases with increasing distance from the centre of the wave.

If we assume that p varies sinusoidally, we may write

$$p(r,t)=\frac{A}{r}Sin\frac{2\pi}{\lambda}(ct-r)=\frac{A}{r}Sin(\omega t-kr)=\frac{A}{r}Sin(\omega t-\vec{k}.\vec{r})$$

or $p(r,t)=\frac{A}{r}Cos\frac{2\pi}{\lambda}(ct-r)=\frac{A}{r}\mathrm{Cos}(\omega t-kr)=\frac{A}{r}\mathrm{Cos}(\omega t-\vec{k}.\vec{r})$

or $p(r,t)=\frac{A}{r}e^{i\frac{2\pi}{\lambda}(ct-r)}=\frac{A}{r}e^{i(\omega t-kr)}=\frac{A}{r}e^{i(\omega t-\vec{k}.\vec{r})}$

if we use the complex form. The pressure amplitude varies inversely as distance the distance.

That p should vary inversely as r in spherical wave can be understood from a simple physical consideration. If the average power W emitted by the source is constant, the intensity at a distance r from the centre of the waves is $I=\frac{W}{4\pi r^2}$. Since $I\propto p^2$, we must have $p\propto\frac{1}{r}$. When p varies sinusoidally we are automatically led to the expression $p(r,t)=\frac{A}{r}Sin(\omega t-kr)$.