Superposition of Harmonic Waves

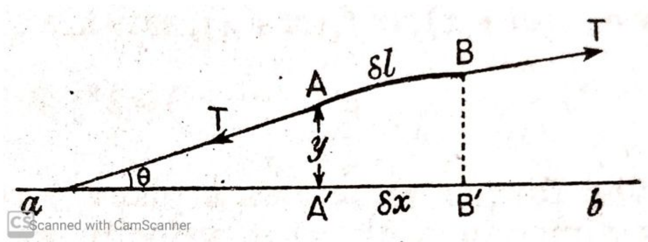
We have seen that in vibration inertia and elasticity are localized, while in wave motion they are distributed. String constitutes a remarkable class by themselves in the sense that it is a vibrator and in the same time a medium for wave propagation. Therefore, the theoretical study of its behaviour is very important.

Strings also have point of much practical interest. In the world of music, stringed instruments play a very important role. Musical sound is produced in them by transverse vibration of strings kept under tension. A point of a string is displaced from its point of rest by plucking, strucking or bowing. The string then vibrates either as a whole or in segments producing musical note of definite pitch and quality. Theoretically a string is defined as ‘a perfectly uniform and perfectly flexible filament of solid matter stretched between two fixed points. Therefore, string is an ideal body which is never realised in practice but closely approximated to by most of the strings employed in musical instruments. The actual string always possesses some amount of rigidity, the effect of which diminishes as its length to diameter ratio increases. The transverse vibration of the ideal string is affected by tension only and not by rigidity.

Transverse waves on a string:

A string under tension resists any tendency to displace any point on it from its position of rest. This resistance supplies the restoring force when the displacement occurs. If a point on a long thin stretched string is displaced at right angles to its length and then released, the point begin to vibrate, and starts a twin displacement wave on either side of it which travel along the string. These waves are reflected from the fixed ends and interfere with the incident waves and forms stationary wave pattern. The string then vibrates in a manner determined by the mode of excitation, the frequency depending on its length, tension and mass per unit length as also the mode of vibration.

Velocity of transverse waves along a string:



Let ab represents a stretched undisplaced string, and AB represents a portion of it in displaced condition. The amplitude of displacement is supposed to be small. Let us take X-axis along the length of the string and Y-axis in the direction of displacement, which is perpendicular to X-axis.

As the string is perfectly flexible, the tension T will be the same throughout the string and acts tangentially at every point on AB. Let the tangent at A make an angle θ with ab. As θ is very small, the transverse component of the tension T (i.e. along AA’) will be

$$TSin\theta \simeq T \tan\theta = T\frac{\partial y}{\partial x}.$$

So, the component of tension at B along B'B will be

$$T\frac{\partial y}{\partial x}+\frac{\partial}{\partial x}(T\frac{\partial y}{\partial x})\delta x\,.$$

Therefore, the net force tending to displace AB is

$$[T\frac{\partial y}{\partial x}+\frac{\partial}{\partial x}(T\frac{\partial y}{\partial x})\delta x]-T\frac{\partial y}{\partial x}=T\frac{\partial^2 y}{\partial x^2}\delta x$$

This force must be equal to mass x acceleration.

$\therefore T\frac{\partial^2 y}{\partial x^2}\delta x=(m\delta x).\frac{\partial^2 y}{\partial t^2}$, where m is the mass per unit length of the string.

$$\therefore \frac{\partial^2 y}{\partial t^2}=\frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

$$\therefore \frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$

where $c=\sqrt{\frac{T}{m}}$ is the wave velocity of the transverse wave in the string. This velocity is independent of the frequency and also the amplitude of vibration (assumed to be small).

Stationary or standing waves:

An interesting and very important type of motion (vibration) is set up in a medium when two progressive waves of the same frequency moving in opposite directions in a medium are superposed. Such vibrations are known as stationary or standing waves. This type of motion occurs when the medium is limited in extent and waves are generated in it. The outgoing waves are reflected at the boundary, and the incident and reflected wave systems by their superposition form the stationary wave pattern.

Case I: Waves of equal amplitude:

Let the two waves moving in opposite directions be represented by

$\xi_1=aSin(\omega t-kx)$ and $\xi_2=aSin(\omega t+kx)$

where ξ_1, ξ_2 are the particle displacements due to the two waves.

The resultant displacement is given by

$$\xi=\xi_1+\xi_2=aSin(\omega t-kx)+aSin(\omega t+kx)$$
$$=2aCoskxSin\omega t=ASin\omega t$$

where $A = 2aCoskx$.

This represents a simple harmonic motion of the same frequency ω as the waves, but of amplitude $A = 2aCoskx$, which depends on the position x of the particle. It is not a progressive wave as the final expression of the phase angle does not contain both x and t .

The amplitude $A = 2aCoskx$ changes as x changes. When $Coskx = 0$, then $A = 0$. This happens when

$$kx = \frac{2\pi}{\lambda}x = (2n-1)\frac{\pi}{2} \text{ where } n = 1,2,3,......etc.$$

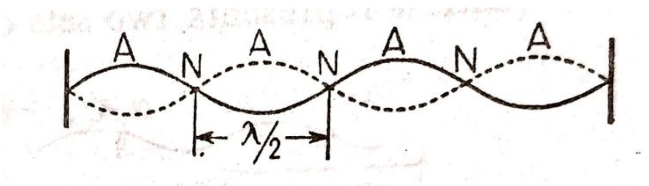
Or, $x = (2n-1)\frac{\lambda}{4}$

$\therefore$ when $n = 1$, $\quad x_1 = \frac{\lambda}{4}$

$n = 2$, $\quad x_2 = \frac{3\lambda}{4}$

$n = 3$, $\quad x_3 = \frac{5\lambda}{4}$ etc.

Such points, where the amplitude of displacement is zero, are called displacement nodes (marked as N in the figure). We can see that the distance between two successive nodes is $\frac{\lambda}{2}$.



When $Coskx = \pm 1$, the amplitude is a maximum and has the value $2a$. This happens when

$$kx = \frac{2\pi}{\lambda}x = n\pi \text{ where } n = 1,2,3,......etc.$$

Or, $x = n\frac{\lambda}{2}$

$\therefore$ when $n = 1$, $\quad x_1 = \frac{\lambda}{2}$

$n = 2$, $\quad x_2 = \frac{2\lambda}{2}$

$n = 3$, $\quad x_3 = \frac{3\lambda}{2}$ etc.

These points of maximum amplitude are called displacement antinodes (marked as A in the figure). The distance between two successive antinodes is too $\frac{\lambda}{2}$. The distance between a node and its nearest antinode is $\frac{\lambda}{4}$. The section between two neighbouring nodes is called a loop.

If through one nodal distance the amplitude is positive, in the next one it becomes negative and vice-versa. Hence the vibration is in reverse direction after each node. Since $\xi = ASin\omega t$, the phase of vibration remains the same until A changes sign. Thus, from one node to the next node the vibrations are in the same phase with changing amplitude; there is an abrupt change of phase by π in the vibrations throughout the next node. Thus, the neighbouring loops vibrate in opposite phases (see fig). With increasing t, $Sin\omega t$ passes through all possible values from +1 to -1 including 0. When $Sin\omega t = 0$, $\xi = 0$ i.e. displacement of all particles are zero i.e. all particles are momentarily in their undisplaced positions. This occurs twice during each vibration. The period of vibration is $T = \frac{2\pi}{\omega}$.

Case-II: Waves of unequal amplitude:

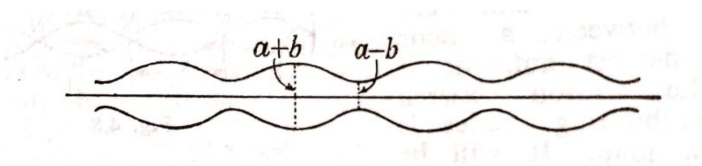
If the plane waves are reflected from a yielding boundary, the reflected waves will have amplitude smaller than the incident waves. The superposition of the two wave systems will form a stationary wave pattern in which there will be some motion at the nodes.

Let the incident wave system be represented by $\xi_1 = aSin(\omega t - kx)$ and the reflected wave system be represented by $\xi_2 = bSin(\omega t + kx)$, where $a > b$.

The resultant displacement is

$$\begin{aligned} \xi &= \xi_1 + \xi_2 = aSin(\omega t - kx) + bSin(\omega t + kx) \\ &= (a+b)CoskxSin\omega t - (a-b)SinkxCos\omega t \end{aligned}$$

This equation represents two sets of stationary waves, both having the same frequency ω as the waves and having phase difference of $\frac{\pi}{2}$. The amplitude varies from point to point and are gives by $(a+b)Coskx$ for one and $(a-b)Sinkx$ for the other.



$(a+b)Coskx$ has maxima at $x = n\frac{\lambda}{2}$ where n=0,1,2,3.......etc and the values of these maxima are all $(a+b)$. The amplitude of the second vibration at these places is zero.

$(a-b)Sinkx$ has maxima at $x=(n+\frac{1}{2})\frac{\lambda}{2}$ where n=0,1,2,3.......etc and the values of these maxima are all $(a-b)$. At these points amplitude due to the first vibration is zero.

We thus see that in the resultant vibration, there are alternate planes of maximum and minimum vibration (see the figure), the ratio of amplitudes is $(a+b):(a-b)$.

Stationary wave in a vibrating string:

(a) Stationary wave in a string fixed at both ends:

The differential wave equation of transverse wave in a string is given by

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$

Solution of this equation can be written as

$y(x,t)=(ACoskx+BSinkx)(CCos\omega t+DSin\omega t)$ ....(1)

Since the string is fixed at $x=0$ and $x=l$, there will be no displacement at these points for all times. Thus

$y(0,t)=0$ ....(a)

and $y(l,t)=0$ ....(b)

Now, applying boundary condition (a) in eq. (1), we have

$y(0,t)=0=A(CCos\omega t+DSin\omega t)$

$\therefore A=0$

$\therefore y(x,t)=BSinkx(CCos\omega t+DSin\omega t)$ ....(2)

Now, applying boundary condition (b) in eq. (2), we have

$y(l,t)=0=BSinkl(CCos\omega t+DSin\omega t)$

$\therefore Sinkl=0$ [$\because B=0$ or $CCos\omega t+DSin\omega t=0$ will make displacement zero for all times. These solutions are trivial and the string does not move at all which we do not want.]

$\therefore Sinkl=Sinn\pi$, where n is an integer

$\therefore kl=n\pi$

$\therefore k=\frac{n\pi}{l}$

Or, $\omega = ck = \frac{n\pi c}{l}$

$\therefore \omega_n = \frac{n\pi c}{l}$ ....(3)

$\therefore \omega_1 = \frac{\pi c}{l}, \omega_2 = \frac{2\pi c}{l}, \omega_3 = \frac{3\pi c}{l}, ...etc.$ Only such values of ω are possible.

The solution can now be written as

$$y_n = B_n Sin\frac{n\pi x}{l}(C_n Cos\frac{n\pi ct}{l} + D_n Sin\frac{n\pi ct}{l}) \text{ ....(4)}$$

Here subscript n has been included, as ω can have an infinite set of values characterized by integral values of n .

The coefficients and displacements for different n values are generally different. Equation (4) can be written as

$$y_n = Sin\frac{n\pi x}{l}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l}) \text{ ....(5)}$$

Here $a_n = B_n C_n$ and $b_n = B_n D_n$ are new single constants.

The general solution is given by

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l})Sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty}(a_n Cos\omega_n t + b_n Sin\omega_n t)Sin\frac{\omega_n x}{c} \end{aligned} \text{ ....(6)}$$

This equation can also be written as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} c_n Sin\frac{n\pi x}{l} Cos(\frac{n\pi ct}{l} - \phi_n) \\ &= \sum_{n=1}^{\infty} c_n Sin\frac{\omega_n x}{c} Cos(\omega_n t - \phi_n) \end{aligned} \text{ ....(7)}$$

where $a_n = c_n Cos\phi_n$ and $b_n = c_n Sin\phi_n$ i.e. $c_n = \sqrt{a_n{}^2 + b_n{}^2}$ and $\tan\phi_n = \frac{b_n}{a_n}$.

Thus, in the n-th mode, an element dx of the string vibrates simple harmonically with amplitude $c_n Sin\frac{\omega_n x}{c}$ and angular frequency ω_n .

(b) Stationary wave in a string free at both ends:

Displacement of any point of the string is given by

$y(x,t) = (ACoskx + BSinkx)(CCos\omega t + DSin\omega t)$ ....(1)

Since the string is free at both ends $x = 0$ and $x = l$, displacements are maximum and strains are zero at these positions.

$\therefore \frac{\partial y}{\partial x} = 0$ at $x = 0$ and $x = l$.

Now, $\frac{\partial y}{\partial x} = (-ASinkx + BCoskx)k(CCos\omega t + DSin\omega t)$ ....(2)

Now, applying the first boundary condition, we get

$\frac{\partial y}{\partial x} = 0 = -Bk(CCos\omega t + DSin\omega t)$

$\therefore B = 0$

Therefore, the eq. (1) and eq. (2) become

$y(x,t) = ACoskx(CCos\omega t + DSin\omega t)$ ....(3)

and $\frac{\partial y}{\partial x} = -AkSinkx(CCos\omega t + DSin\omega t)$ ....(4)

Now, we apply the second boundary condition in eq. (4) and get

$\frac{\partial y}{\partial x} = 0 = -AkSinkl(CCos\omega t + DSin\omega t)$

$\therefore Sinkl = 0$

$\therefore kl = n\pi$, where n is an integer.

$\therefore k = \frac{n\pi}{l}$

Or, $\omega = ck = \frac{n\pi c}{l}$

$\therefore \omega_n = \frac{n\pi c}{l}$

$\therefore \omega_1 = \frac{\pi c}{l}, \omega_2 = \frac{2\pi c}{l}, \omega_3 = \frac{3\pi c}{l}, ...etc.$ Only such values of ω are possible.

The solution can now be written as

$$y_n = A_n Cos\frac{n\pi x}{l}(C_n Cos\frac{n\pi ct}{l} + D_n Sin\frac{n\pi ct}{l})$$

Here subscript n has been included, as ω can have an infinite set of values characterized by integral values of n.

$$y_n = Cos\frac{n\pi x}{l}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l}) \quad\text{....................................(5)}$$

Here $a_n = A_n C_n$ and $b_n = A_n D_n$ are new single constants.

The general solution is given by

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l})Cos\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty}(a_n Cos\omega_n t + b_n Sin\omega_n t)Cos\frac{\omega_n x}{c} \end{aligned} \quad\text{....................................(6)}$$

This equation can also be written as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} c_n Cos\frac{n\pi x}{l} Cos(\frac{n\pi ct}{l} - \phi_n) \\ &= \sum_{n=1}^{\infty} c_n Cos\frac{\omega_n x}{c} Cos(\omega_n t - \phi_n) \end{aligned} \quad\text{....................................(7)}$$

where $a_n = c_n Cos\phi_n$ and $b_n = c_n Sin\phi_n$ i.e. $c_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$ and $\tan\phi_n = \frac{b_n}{a_n}$.

Thus, in the n-th mode, an element dx of the string vibrates simple harmonically with amplitude $c_n Cos\frac{\omega_n x}{c}$ and angular frequency ω_n.

Characteristic frequencies:

The characteristic frequencies with which a string stretched between two rigid supports at $x = 0$ and $x = l$ vibrates are given by

$$\omega_n = \frac{n\pi c}{l}$$

The frequency f_n of the n-th mode of vibration is $f_n = \frac{\omega_n}{2\pi}$.

$$\therefore f_n = \frac{\omega_n}{2\pi} = \frac{n\pi c}{l}.\frac{1}{2\pi} = \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{T}{m}}$$

where n is an integer.

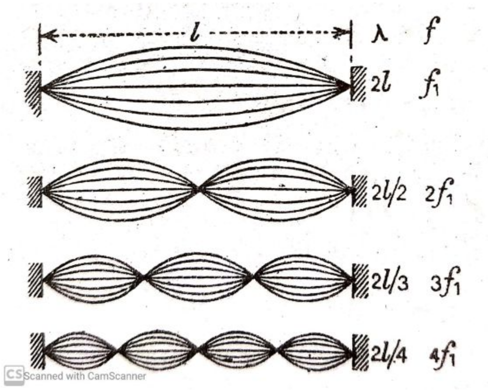
The lowest frequency corresponds to $n=1$, and is called the fundamental frequency. It has the value

$$f_1 = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

Since wavelength=velocity/frequency, the wavelength λ in this case is $2l$. Other frequencies are integral multiples of this value and the wavelength n times shorter.

Modes of vibration of a string fixed at both ends:

The characteristic frequencies of a string can also be derived from simple physical considerations. As soon as a point of the string is disturbed, two waves of same amplitude, both moving with the same velocity $c=\sqrt{\frac{T}{m}}$, start from the point towards the ends. Since the string is rigidly fixed at both ends, no displacement can occur there and the displacement waves are fully reflected from the ends. The superposition of two identical but moving in opposite direction wave trains produces a stationary wave system of the same frequency as the component waves. As both ends are rigidly fixed and no displacement can occur there, two nodes will be created at the ends. Only those modes of vibration persist which produce nodes at two ends. As there may be any number of nodes in between, various wavelengths, and hence frequencies of vibration, are possible.



If l be the length of the string then the simplest mode of vibration occurs when the string vibrates as a whole (top one in the figure). Then, $l=\frac{\lambda_1}{2}$ i.e. $\lambda_1 = 2l$, where λ_1 is the longest possible wavelength. Next mode of vibration occurs when the string vibrates in two segments (second from top). The corresponding wavelength is $\lambda_2 = l$. In the third mode, the string vibrates in three segments such that $l = 3\frac{\lambda_3}{2}$ and the corresponding wavelength is $\lambda_3 = \frac{2l}{3}$. In the next mode of vibration, the string vibrates in four segments and $l = 4\frac{\lambda_4}{2}$ i.e. $\lambda_4 = \frac{l}{2}$.

Since velocity = frequency x wavelength, the corresponding frequencies are given by

$$f_1 = \frac{c}{2l} = \frac{1}{2l}\sqrt{\frac{T}{m}},$$

$$f_2 = \frac{c}{2l/2} = 2.\frac{1}{2l}\sqrt{\frac{T}{m}} = 2f_1,$$

$$f_3 = \frac{c}{2l/3} = 3.\frac{1}{2l}\sqrt{\frac{T}{m}} = 3f_1,$$

$$f_4 = \frac{c}{2l/4} = 4.\frac{1}{2l}\sqrt{\frac{T}{m}} = 4f_1 \quad \text{etc.}$$

In general, $f_n = \frac{n}{2l}\sqrt{\frac{T}{m}}$.

The string is therefore capable of emitting a full harmonic series, i.e. a fundamental (f_1) and all its multiples. When the string vibrates in one segment, the frequency emitted by it under a given tension is the lowest and the note is called the fundamental. When it vibrates in two segments, the frequency is doubled and the corresponding note is called second harmonic or first overtone. In the third mode of vibration the frequency is thrice that of the fundamental and the note is called third harmonic or second overtone, and so on. In general, if the string vibrates in n segments then the frequency of vibration is n times that of the fundamental and the corresponding note is the n-th harmonic.

A vibrating string represents a stationary wave:

When any portion of a stretched string is disturbed transversely, the disturbance travels along the string satisfying the differential wave equation given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The general solution of this equation is of the form

$$y(x,t) = F_1(ct - x) + F_2(ct + x)$$

Where F_1 and F_2 are two arbitrary functions. Assuming the functions to be simple harmonic, we may write

$$y(x,t) = a_1 Sin(\omega t + kx) + a_2 Sin(\omega t - kx)$$

Two terms represent oppositely moving waves of amplitudes a_1 and a_2. The waves are subject to the boundary conditions that the value of displacement y at the two ends must be zero at all times, since the string is fixed at the ends.

At $x=0$, $y=a_1 Sin\omega t+a_2 Sin\omega t=0$ for all values of t.

This gives $a_2=-a_1=\frac{1}{2}C$ (say)

$$\therefore y=\frac{1}{2}CSin(\omega t+kx)-\frac{1}{2}CSin(\omega t-kx)$$

At $x=l$, we have

$$y=\frac{1}{2}CSin(\omega t+kl)-\frac{1}{2}CSin(\omega t-kl)=0 \text{ for all t.}$$

$$\therefore y=CSinklCos\omega t=0 \text{ for all t.}$$

Hence, we must have

$kl=n\pi$, where n is an integer.

$$\therefore k=\frac{n\pi}{l}$$

$$\therefore \omega=ck=\frac{n\pi c}{l}$$

$$\therefore \omega_n=\frac{n\pi c}{l}$$

The values of k depends on n. Imposition of boundary conditions allows only certain values of frequency. The value of C will, in general, also depend on n.

$$\therefore \begin{aligned} y_n &= \frac{1}{2}CSin(\omega_n t+\frac{n\pi x}{l})-\frac{1}{2}CSin(\omega_n t-\frac{n\pi x}{l}) \\ &= C_n SinkxCos\omega_n t=C_n Sin\frac{n\pi x}{l}Cos\omega_n t \end{aligned}$$

$$\therefore y_n=C_n Sin\frac{n\pi x}{l}Cos\frac{n\pi ct}{l}$$

This equation gives the displacement of the string at point x and time t when it is vibrating in its n-th mode. The vibration is due to the superposition of the two oppositely moving waves $\frac{1}{2}CSin(\omega_n t+\frac{n\pi x}{l})$ and $\frac{1}{2}CSin(\omega_n t-\frac{n\pi x}{l})$ where $\omega_n=\frac{n\pi c}{l}$. The amplitude $C_n Sin\frac{n\pi x}{l}$ of the resultant vibration varies with position x, and reaches a maximum (anti-node) when $\frac{n\pi x}{l}=\frac{\pi}{2}$ or $x=\frac{l}{2n}$ and minimum (zero value, node) when $\frac{n\pi x}{l}=\pi$ or $x=\frac{l}{n}$. At $x=0$, it is always zero. In the n-th mode the string then vibrates in n segments. The neighbouring segments are in opposite phases i.e. when one segment moves up, the neighbouring segment moves down.

The value of C_n depends on the energy with which the motion is initiated.

The string has natural frequencies of value ω_n and is maintained in vibration by waves of frequency ω_n.

Values of coefficients a_n and b_n :

The displacement of any portion of a vibrating string is given by

$$y(x,t) = \sum_{n=1}^{\infty}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l})Sin\frac{n\pi x}{l} \text{ ....(1)}$$

At initial time of vibration (t=0) eq. (1) becomes

$$y(x,0) = \sum_{n=1}^{\infty} a_n Sin\frac{n\pi x}{l} \text{ ....(2)}$$

Differentiating eq. (1) with respect to time t, we get

$$\dot{y}(x,t) = \sum_{n=1}^{\infty}(-a_n Sin\frac{n\pi ct}{l} + b_n Cos\frac{n\pi ct}{l})\frac{n\pi c}{l}Sin\frac{n\pi x}{l} \text{ ....(3)}$$

At the initial time of vibration eq. (3) becomes

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} b_n\frac{n\pi c}{l}Sin\frac{n\pi x}{l} \text{ ....(4)}$$

Multiplying equations (2) and (4) by $Sin\frac{m\pi x}{l}$ and integrating between the limits $x=0$ and $x=l$, we get

$$\int_{x=0}^{l} y(x,0)Sin\frac{m\pi x}{l}dx = \int_{x=0}^{l}\sum_{n=1}^{\infty} a_n Sin\frac{n\pi x}{l}Sin\frac{m\pi x}{l}dx \text{ ....(5)}$$

$$\int_{x=0}^{l} \dot{y}(x,0)Sin\frac{m\pi x}{l}dx = \int_{x=0}^{l}\sum_{n=1}^{\infty} b_n\frac{n\pi c}{l}Sin\frac{n\pi x}{l}Sin\frac{m\pi x}{l}dx \text{ ....(6)}$$

Now, since $\int_{x=0}^{l} Sin\frac{n\pi x}{l}Sin\frac{m\pi x}{l}dx$ = 0 for $m \neq n$

= $\frac{l}{2}$ for $m = n$

$$= \frac{l}{2}\delta_{mn}$$

Equations (5) and (6) become

$$a_n = \frac{2}{l}\int_{x=0}^{l} y(x,0) Sin\frac{n\pi x}{l}dx \text{ ....(7)}$$

$$b_n = \frac{2}{n\pi c}\int_{x=0}^{l} \dot{y}(x,0) Sin\frac{n\pi x}{l}dx \text{ ....(8)}$$

Energy of a vibrating string:

Let us consider a uniform stretched string of length l, mass per unit length m, under tension T and fixed at both ends i.e. $x=0$ and $x=l$. We also assume that a transverse vibration has been set up in the string.

Its general displacement formula is given by

$$y(x,t) = \sum_{n=1}^{\infty}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l})Sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} c_n Sin\frac{n\pi x}{l} Cos(\frac{n\pi ct}{l} - \phi_n)$$

$$= \sum_{n=1}^{\infty} \psi_n(t) Sin\frac{n\pi x}{l}$$

where $a_n = c_n Cos\phi_n$ and $b_n = c_n Sin\phi_n$ i.e. $c_n = \sqrt{a_n{}^2 + b_n{}^2}$ and $\tan\phi_n = \frac{b_n}{a_n}$, and

$\psi_n(t) = (a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l}) = c_n Cos(\frac{n\pi ct}{l} - \phi_n)$.

Kinetic energy:

Now, velocity at any instant is given by

$$\dot{y}(x,t) = \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\dot{\psi}_n(t) Sin\frac{n\pi x}{l}$$

Let us consider an element of length δx of the string. Its kinetic energy at any instant t is

$$\delta E_k = \frac{1}{2}(m\delta x)(\frac{\partial y}{\partial t})^2 = \frac{1}{2}(m\delta x)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\psi}_n(t) Sin\frac{n\pi x}{l}\dot{\psi}_m(t) Sin\frac{m\pi x}{l}$$

$\therefore$ Total kinetic energy of the string is given by

$$E_k = \frac{1}{2}m\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\psi}_n(t)\dot{\psi}_m(t)\int_{x=0}^{l} Sin\frac{n\pi x}{l}Sin\frac{m\pi x}{l}dx$$

$$= \frac{1}{2}m\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\psi}_n(t)\dot{\psi}_m(t)\frac{l}{2}\delta_{mn}$$,where $\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ for $m \neq n$

$$= \frac{ml}{4}\sum_{n=1}^{\infty}[\dot{\psi}_n(t)]^2 = \frac{M}{4}\sum_{n=1}^{\infty}[\dot{\psi}_n(t)]^2$$

where $M = ml$ is the total mass of the string.

Potential energy:

The potential energy may be calculated in the following way:

The force acting on an element of length δx when displaced by y is $T\frac{\partial^2 y}{\partial x^2}\delta x$.

If we displace the element further by distance δy, work done is $\delta W = (T\frac{\partial^2 y}{\partial x^2}\delta x)\delta y$. Total work done in displacing the element by y is stored in the string as potential energy and is given by

$$\delta E_p = \int_0^y (T\frac{\partial^2 y}{\partial x^2}\delta x)dy.$$

$\therefore$ Potential energy of the whole string is

$$E_p = \int_{x=0}^{l}\int_0^y (T\frac{\partial^2 y}{\partial x^2})dydx$$

$$= \int_0^y\int_{x=0}^{l} T\frac{\partial}{\partial x}(\frac{\partial y}{\partial x})dxdy = \int_{x=0}^{l}\int_0^{\frac{\partial y}{\partial x}} T\frac{\partial}{\partial x}(\frac{\partial y}{\partial x})dx\frac{\partial y}{\partial x}dx$$

$$= \int_{x=0}^{l}\int_0^{\frac{\partial y}{\partial x}} T\frac{\partial y}{\partial x}\frac{\partial}{\partial x}(\frac{\partial y}{\partial x})dxdx = \int_{x=0}^{l}\int_0^{\frac{\partial y}{\partial x}} T\frac{\partial y}{\partial x}\partial(\frac{\partial y}{\partial x})dx$$

$$= \int_0^l \frac{T}{2}(\frac{\partial y}{\partial x})^2 dx$$

Now, $y(x,t) = \sum_{n=1}^{\infty}\psi_n(t)Sin\frac{n\pi x}{l}$

$$\therefore \frac{\partial y}{\partial x} = \sum_{n=1}^{\infty}\psi_n(t)\frac{n\pi}{l}Cos\frac{n\pi x}{l}$$

$$\therefore (\frac{\partial y}{\partial x})^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\psi_n(t)\frac{n\pi}{l}Cos\frac{n\pi x}{l}\psi_m(t)\frac{m\pi}{l}Cos\frac{n\pi x}{l}$$

$$E_p = \int_0^l \frac{T}{2}(\frac{\partial y}{\partial x})^2 dx = \frac{T}{2}\int_0^l \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\psi_n(t)\frac{n\pi}{l}Cos\frac{n\pi x}{l}\psi_m(t)\frac{m\pi}{l}Cos\frac{m\pi x}{l}dx$$

$$= \frac{T}{2}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\psi_n(t)\psi_m(t)\frac{n\pi}{l}\frac{m\pi}{l}\int_0^l Cos\frac{n\pi x}{l}Cos\frac{m\pi x}{l}dx$$

$$= \frac{T}{2}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\psi_n(t)\psi_m(t)\frac{n\pi}{l}\frac{m\pi}{l}\frac{l}{2}\delta_{mn}$$

$$\therefore = \frac{T}{2}.\frac{l}{2}\sum_{n=1}^{\infty}[\frac{n\pi}{l}\psi_n(t)]^2$$

$$= \frac{Tl}{4}\sum_{n=1}^{\infty}[\psi_n(t)]^2(\frac{n\pi}{l})^2$$

$$= \frac{c^2 ml}{4}\sum_{n=1}^{\infty}[\psi_n(t)]^2(\frac{n\pi}{l})^2$$

$$= \frac{M}{4}\sum_{n=1}^{\infty}[\psi_n(t)]^2(\frac{n\pi c}{l})^2$$

$\therefore$ Total energy of the whole string is given by

$$E = E_k + E_p = \frac{M}{4}\sum_{n=1}^{\infty}[\dot{\psi}_n(t)]^2 + \frac{M}{4}\sum_{n=1}^{\infty}[\psi_n(t)]^2(\frac{n\pi c}{l})^2$$

$$= \frac{M}{4}\sum_{n=1}^{\infty}\{[\dot{\psi}_n(t)]^2 + [\psi_n(t)]^2(\frac{n\pi c}{l})^2\}$$

Now, $\psi_n(t) = (a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l})$

$$\therefore \dot{\psi}_n(t) = (-a_n Sin\frac{n\pi ct}{l} + b_n Cos\frac{n\pi ct}{l})\frac{n\pi c}{l}$$

$$\therefore [\dot{\psi}_n(t)]^2 + [\psi_n(t)]^2.(\frac{n\pi c}{l})^2 = (\frac{n\pi c}{l})^2(a_n{}^2 + b_n{}^2) = (\frac{n\pi c}{l})^2 c_n{}^2$$

$\therefore$ Total energy of the string at any instant t is

$$E = \frac{M}{4}\sum_{n=1}^{\infty}(\frac{n\pi c}{l})^2(a_n{}^2 + b_n{}^2) = \frac{M}{4}\sum_{n=1}^{\infty}\omega_n{}^2(a_n{}^2 + b_n{}^2)$$

$$= \frac{M}{4}.4\pi^2\sum_{n=1}^{\infty}f_n{}^2(a_n{}^2 + b_n{}^2)$$

$$= M\pi^2\sum_{n=1}^{\infty}f_n{}^2(a_n{}^2 + b_n{}^2) = M\pi^2\sum_{n=1}^{\infty}f_n{}^2 c_n{}^2$$

Thus, we see that the total energy of the string is independent of time i.e. total energy is conserved.

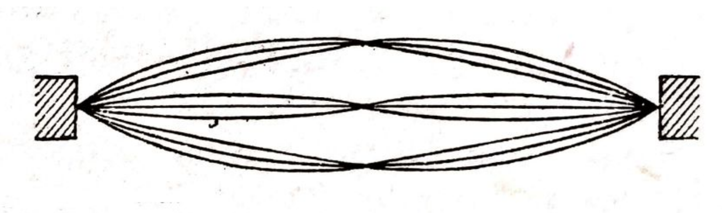
Normal coordinates of a string:

The expression $E = \frac{M}{4}\sum_{n=1}^{\infty}\{[\dot{\psi}_n(t)]^2 + [\psi_n(t)]^2\omega_n^{\ 2}\}$ for total energy shows that it is made up of a sum of terms each of which relates to one mode only. Here $\psi_n(t)$ is called the normal coordinate of a vibrating string. A string may be built up of an infinite number of particle oscillators which are strongly coupled together. So it will have an infinite number of modes of vibration and an infinite number of normal coordinates. The infinite number of ω_n and ψ_n values are the normal frequencies and normal coordinates of the coupled vibration of the string.

Actual vibration of a string:

It is very difficult to excite only one of the many modes of vibrations. The best way of doing it is by resonance with the appropriate frequency.

In general, when we excite a string by plucking or striking or bowing, its vibration is composed of several of the various possible modes of vibration. One particular mode may be predominant, but there will be several other modes of which the string is capable to vibrate under the conditions of excitation. The quality of the emitted note depends on the particular harmonics present and their relative amplitudes.



In a complex mode of vibration, the motion of any point on the string is the resultant of all the motions present appropriate to the component modes present. In the figure below, we have shown a string vibrating with the fundamental and the first overtone. The string vibrates with a node at the centre, with frequency $2f$, it also moves ups and down with the fundamental frequency f and a node at each end.

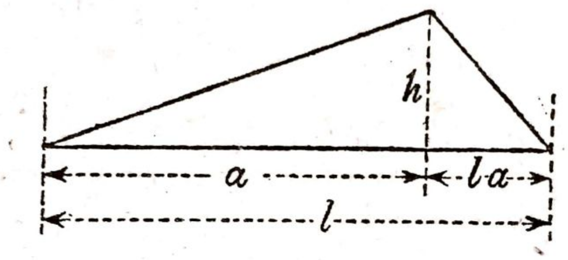
The modes which may be present in a given case are determined by the method of excitation. If a string is pulled up at a point $\frac{1}{p}$ of its length from one end, where p is an integer, no modes of vibration can be present for which the given point is a node. Hence, if f be the fundamental frequency, none of the vibrations can be present for which the frequency is the integral multiple of pf. That is, the p-th, $2p$-th, $3p$ etc. harmonics will absent. Thus plucking in the middle will remove all the even harmonics.

Suppose the vibrating string is touched at some other point with a light object such as a feather or a folded paper. This will restrict all vibrations except those which have nodes at the point touched. For example, a string originally touched at the middle. This will keep all the odd harmonics i.e. 1st, 3rd, 5th, 7th etc. In addition, if the string is touched at one third of the length, only the 3rd, 9th, 15th, 21st etc., harmonics will remain.

Young-Helmholtz law:

Young-Helmholtz law states that when any point of the stretched string is plucked, strucked or bowed, all overtones requiring that point for a node, will be absent in the vibration.

Plucked string:



Let us consider a string of length l, mass per unit length m, fixed at two end i.e. at $x=0$ and $x=l$. The string is stretched at a tension T. So the equation of motion for transverse vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \text{ where } c = \sqrt{\frac{T}{m}}.$$

The solution for general displacement is

$$y(x,t) = \sum_{n=1}^{\infty} (a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l}) Sin\frac{n\pi x}{l}$$

If the string is excited by plucking i.e. the string is given an initial displacement which is maximum at $x=a$, the initial displacement at that point is $y=h$ and the string is released i.e. initial velocity is zero everywhere.

Hence, at $t=0$ the displacement at any point is

$$y(x,0) = \frac{h}{a}x \text{ for } 0<x<a$$

$$= \frac{h}{l-a}(l-x) \text{ for } a<x<l$$

Now,

$$a_n = \frac{2}{l}\int_{x=0}^{l} y(x,0)Sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}[\int_{x=0}^{a}\frac{h}{a}xSin\frac{n\pi x}{l}dx + \int_{x=a}^{l}\frac{h}{l-a}(l-x)Sin\frac{n\pi x}{l}dx]$$

$$= \frac{2}{l}.\frac{h}{a}\int_{x=0}^{a} xSin\frac{n\pi x}{l}dx + \frac{2}{l}.\frac{h}{l-a}\int_{x=a}^{l}(l-x)Sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}.\frac{h}{a}[x.(-\frac{l}{n\pi}Cos\frac{n\pi x}{l}) - 1.(-\frac{l^2}{n^2\pi^2}Sin\frac{n\pi x}{l})]\Big|_0^a$$

$$+\frac{2}{l}.\frac{h}{l-a}[(l-x).(-\frac{l}{n\pi}Cos\frac{n\pi x}{l}) - (-1).(-\frac{l^2}{n^2\pi^2}Sin\frac{n\pi x}{l})]\Big|_a^l \ [\text{Integrating by Kronecker method}]$$

$$= \frac{2}{l}.\frac{h}{a}[-a\frac{l}{n\pi}Cos\frac{n\pi a}{l} + \frac{l^2}{n^2\pi^2}Sin\frac{n\pi a}{l}] + \frac{2}{l}.\frac{h}{l-a}[(l-a)\frac{l}{n\pi}Cos\frac{n\pi a}{l} + \frac{l^2}{n^2\pi^2}Sin\frac{n\pi a}{l}]$$

$$= \frac{2}{l}[-\frac{hl}{n\pi}Cos\frac{n\pi a}{l} + \frac{hl^2}{an^2\pi^2}Sin\frac{n\pi a}{l} + \frac{hl}{n\pi}Cos\frac{n\pi a}{l} + \frac{hl^2}{(l-a)n^2\pi^2}Sin\frac{n\pi a}{l}]$$

$$= \frac{2}{l}\frac{hl^2}{n^2\pi^2}[\frac{1}{a} + \frac{1}{l-a}]Sin\frac{n\pi a}{l}$$

$$a_n = \frac{2}{l}\frac{hl^2}{n^2\pi^2}\frac{l-a+a}{a(l-a)}Sin\frac{n\pi a}{l}$$

$$\therefore = \frac{2}{l}\frac{hl^2}{n^2\pi^2}\frac{l}{a(l-a)}Sin\frac{n\pi a}{l}$$

$$= \frac{2hl^2}{a(l-a)n^2\pi^2}Sin\frac{n\pi a}{l}$$

Again, $b_n = \frac{2}{n\pi c}\int_{x=0}^{l}\dot{y}(x,0)Sin\frac{n\pi x}{l}dx = 0 \quad [\because \dot{y}(x,0) = 0]$.

Thus the formula for general displacement becomes

$$y(x,t) = \sum_{n=1}^{\infty}(a_n Cos\frac{n\pi ct}{l} + b_n Sin\frac{n\pi ct}{l})Sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}(\frac{2hl^2}{a(l-a)n^2\pi^2}Sin\frac{n\pi a}{l}Sin\frac{n\pi x}{l})Cos\frac{n\pi ct}{l}$$

$$= \frac{2hl^2}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}Sin\frac{n\pi a}{l}Sin\frac{n\pi x}{l}Cos\frac{n\pi ct}{l}$$

$$= \frac{2hl^2}{a(l-a)\pi^2}[Sin\frac{\pi a}{l}Sin\frac{\pi x}{l}Cos\frac{\pi ct}{l} + \frac{1}{4}Sin\frac{2\pi a}{l}Sin\frac{2\pi x}{l}Cos\frac{2\pi ct}{l} + \frac{1}{9}Sin\frac{3\pi a}{l}Sin\frac{3\pi x}{l}Cos\frac{3\pi ct}{l} + ...]$$

So the displacement for the n-th harmonic is

$$y_n(x,t) = \frac{2hl^2}{a(l-a)\pi^2}\frac{1}{n^2} Sin\frac{n\pi a}{l} Sin\frac{n\pi x}{l} Cos\frac{n\pi ct}{l}$$

$$= \frac{A}{n^2} Sin\frac{n\pi a}{l} Sin\frac{n\pi x}{l} Cos\frac{n\pi ct}{l}$$

where $A = \frac{2hl^2}{a(l-a)\pi^2}$ is the same for all modes.

Thus the maximum amplitude for the n-th mode is proportional to $\frac{1}{n^2}$ i.e. the amplitude of higher harmonics falls off rapidly.

Position of nodes:

At the position of the nodes, $y = 0$ for all time t.

$$\therefore Sin\frac{n\pi x}{l} = 0$$

$\therefore \frac{n\pi x_n}{l} = m\pi$, where m is a positive integer.

$\therefore x_n = \frac{ml}{n}$, provided $\frac{m}{n} \leq 1$.

Missing modes:

If $a = \frac{1}{q}l$, where q is an integer, we shall have, $Sin\frac{n\pi a}{l} = Sin\frac{n\pi}{q}$. So, when $\frac{n}{q} = p$, i.e. $n = pq$, where p is an integer, then $Sin\frac{n\pi a}{l} = Sin\frac{n\pi}{q} = Sinp\pi = 0$, i.e. the term $Sin\frac{n\pi a}{l}$ will vanish. This means disappearance of components of vibration for which $n = pq$ (i.e. $n = q, 2q, 3q, etc.$). If $q = 2$ i.e. the string is plucked at the middle of it, then the 2nd, 4th, 6th,8th etc., in fact all even harmonics will be absent. Note that, these harmonics have a node at $a = \frac{1}{q}l$. Hence the statement is "In case of a plucked string, harmonics which have a node at the point of plucking will be absent". This is known as Young's law.