Bessel Equation

Bessel functions are named after the German astronomer Friedrich Wilhelm **Bessel**, who studied them in 1817. The following differential equation is known as the Bessel Differential Equation :

$$\mathbf{x^2\, d^2y/dx^2 + x\, dy/dx + (x^2 - \alpha^2)\, y = 0} \quad \text{----- (1)}$$

Dividing by x^2, the equation may be re-written as :

$$d^2y/dx^2 + (1/x)\, dy/dx + (1 - \alpha^2/x^2)\, y = 0, \quad \text{---- (2)}$$

which shows that the coefficient of dy/dx has a simple pole and that of y has a second order pole at **x = 0**. This is a **regular singular point,** hence we can apply Frobenius' method (named after the **German mathematician Ferdinand Georg Frobenius**) to solve this differential equation about x = 0.

So, we can expect a solution of the form :

$$y = \Sigma\, C_n x^{n+s} \Rightarrow\ dy/dx = \Sigma\, (n+s)\, C_n x^{n+s-1}$$

$$\Rightarrow\ d^2y/dx^2 = \Sigma\, (n+s)(n+s-1)\, C_n x^{n+s-2}$$

Substituting in the differential equation (2) :

$$\Sigma\, (n+s)(n+s-1)\, C_n x^{n+s-2} + \Sigma\, (n+s)\, C_n x^{n+s-2} + \Sigma\, C_n x^{n+s} - \alpha^2\, \Sigma\, C_n x^{n+s-2} = 0$$

$$\Rightarrow\ \Sigma\, [(n+s)^2 - \alpha^2]\, C_n x^{n+s-2} + \Sigma\, C_n x^{n+s} = 0 \quad \text{----- (3)}$$

Since this is an identity, the coefficient of each power of x must separately vanish.

Equating the coefficient of the lowest power of x, i.e., $x^{s-2} \to 0$, we have :

$$\Rightarrow\ [s^2 - \alpha^2]\, C_0 = 0 \Rightarrow\ s = \pm\,\alpha, \text{ or, } C_0 = 0 \quad \text{----- (4)}$$

Similarly, equating the coefficient of $x^{s-1} \to 0$, we have :

$$\Rightarrow\ [(s+1)^2 - \alpha^2]\, C_1 = 0 \Rightarrow\ s = \pm\,\alpha - 1, \text{ or, } C_1 = 0 \quad \text{----- (5)}$$

<u>For s = α,</u>

our eqn. (3) reduces to :

$$\Sigma\, [(n+\alpha)^2 - \alpha^2]\, C_n x^{n+\alpha-2} + \Sigma\, C_n x^{n+\alpha} = 0$$

$$\Rightarrow\ \Sigma\, n\,(n+2\alpha)\, C_n x^{n+\alpha-2} = -\,\Sigma\, C_n x^{n+\alpha} \quad \text{----- (6)}$$

Equating the coefficient of $x^{m+\alpha} \to 0$, we have :

$$(m+2)(m+2+2\alpha)\, C_{m+2} = -\ C_m$$

$$\Rightarrow\ \mathbf{C_{m+2} = -\,C_m / (m+2)(m+2+2\alpha)} \quad \text{----- (7)}$$

Eqn. (7) is known as the **recursion relation**, which allows us to express C_2, C_4, etc., in terms of C_0. Thus, we obtain :

$$C_2 = -\,C_0 / 2\,(2+2\alpha) = -\,C_0 / [2^2\,(1+\alpha)],$$

$$C_4 = -\,C_2 / 4\,(4+2\alpha) = -\,C_2 / [2^2 \times 2 \times (2+\alpha)]$$

$$= C_0 / [2^4 \times 1 \times 2 \times (1+\alpha)(2+\alpha)]$$

$$C_6 = -\,C_4 / 6\,(6+2\alpha) = -\,C_4 / [2^2 \times 3 \times (3+\alpha)]$$

$$= -\ C_0 / [2^6 \times 1 \times 2 \times 3 \times (1+\alpha)(2+\alpha)(3+\alpha)], \text{ etc.}$$

$$\mathbf{C_{2n} = (-)^n\ C_0 / [2^{2n} \times n! \times (1+\alpha)(2+\alpha)\cdots(n+\alpha)]},$$

<u>For integral values of **α** :</u> $1/[(1+\alpha)(2+\alpha)\cdots(n+\alpha)]$

$$= 1 \times 2 \times \cdots \alpha / [1 \times 2 \times \cdots \alpha \times (1+\alpha)(2+\alpha)\cdots(n+\alpha)]$$

$$= \alpha! / (n+\alpha)!$$

$$\Rightarrow\ \mathbf{C_{2n} = (-)^n\ \alpha\,!\, C_0 / 2^{2n} \times n!\,(n+\alpha)}\,!$$

For non-integral values of α :

We know that : $\Gamma(n+\alpha+1) = (n+\alpha)\,\Gamma(n+\alpha) = (n+\alpha)(n+\alpha-1)\,\Gamma(n+\alpha-1)$

$= (n+\alpha)(n+\alpha-1)\cdots(2+\alpha)\,(1+\alpha)\,\Gamma(\alpha+1)$

$\Rightarrow (1+\alpha)(2+\alpha)\cdots(n+\alpha) = \Gamma(n+\alpha+1)/\,\Gamma(\alpha+1)$

$\Rightarrow \mathbf{C_{2n} = (-)^n\ \Gamma(\alpha+1)\, C_0 / 2^{2n} \times n!\, \Gamma(n+\alpha+1)}$

So, the solution for $\mathbf{s = \alpha}$ is :

$$y_1 = \Sigma\, C_{2n}\, x^{2n+\alpha} = x^{\alpha}\, \Sigma\, C_{2n}\, x^{2n} = C_0\, x^{\alpha}\ \Gamma(\alpha+1)\ \Sigma\, (-)^n\ [1/n!\ \Gamma(n+\alpha+1)]\ (x/2)^{2n}$$

The constant $\Gamma(\alpha+1)$ may be **absorbed** in C_0, so that y_1 may be written as :

$$\mathbf{y_1 = C_0'\, x^{\alpha}\ \Sigma\, (-)^n\, [1/n!\ \Gamma(n+\alpha+1)]\ (x/2)^{2n}}$$

Another solution may be obtained with the choice $\mathbf{s = -\alpha}$:

Just replacing ‘α’ by ‘$-\alpha$’, we get

$$\mathbf{y_2 = C_0''\, x^{-\alpha}\ \Sigma\, (-)^n\, [1/n!\ \Gamma(n-\alpha+1)]\ (x/2)^{2n}}$$

and the general solution may be written as $y = y_1 + y_2$

Generating Function

The generating function for Bessel function is : $\mathbf{G(x, t) = e^{x/2\,(t-1/t)}}$,

which is expanded as : $\mathbf{\Sigma_n\, t^n\, J_n(x)}$, where ‘n’ runs from $-\infty$ **to** $+\infty$.

$G(x, t)$ may be written as : $e^{xt/2}\, e^{-x/2t} = \Sigma_r\, (xt/2)^r/r! \times \Sigma_s\, (-x/2t)^s/s!$,

where both indices ‘r’ and ‘s’ runs from **0 to** $+\infty$.

So, $G(x, t) = \Sigma_r\, \Sigma_s\, (x/2)^{r+s} \times (t)^{r-s} \times (-1)^s/r!s!$

Put : $r - s = n \Rightarrow r + s = n + 2s$

$\Rightarrow G(x, t) = \Sigma_n\, \Sigma_s\, (x/2)^{n+2s} \times (t)^n \times (-1)^s/(n+s)!\ s!$

$= \Sigma_n\, (t)^n\, (x/2)^n\, \Sigma_s\, (x/2)^{2s} \times (-1)^s/(n+s)!\ s!$

Note that ‘n’ runs from $-\infty$ **to** $+\infty$,

By definition, this equals : $\Sigma_n\, t^n\, J_n(x)$. So, equating the coefficient of t^n on both sides,

$J_n(x) = (x/2)^n\, \Sigma_s\, (x/2)^{2s} \times (-1)^s/(n+s)!\ s!$,

which clearly agrees with the expression obtained from solving the differential equation.

Recursion Relations

$G(x, t) = e^{x/2\,(t-1/t)} = \Sigma_n\, t^n\, J_n(x)$

Differentiating both the expressions partially w.r.t. ‘x’ :

$\tfrac{1}{2}\,(t - 1/t)\, \mathbf{e^{x/2\,(t-1/t)}} = \Sigma_n\, t^n\, J_n'(x)$

Replacing the expression $[e^{x/2\,(t-1/t)}]$ on the LHS again by $\Sigma_n\, t^n\, J_n(x)$:

$\tfrac{1}{2}\,(t - 1/t)\, \mathbf{\Sigma_n\, t^n\, J_n(x)} = \Sigma_n\, t^n\, J_n'(x)$

$\Rightarrow \tfrac{1}{2}\, \Sigma_n\, t^{n+1}\, J_n(x) - \tfrac{1}{2}\, \Sigma_n\, t^{n-1}\, J_n(x) = \Sigma_n\, t^n\, J_n'(x)$

Equating the coefficient of t^m on both sides :

$$[J_{m-1}(x) - J_{m+1}(x)] = 2J_m'(x) \quad \text{---- (1)}$$

Starting again from the definition : $G(x, t) = e^{x/2\,(t - 1/t)} = \Sigma_n t^n J_n(x)$,
differentiating both the expressions partially w.r.t. ‘t’ :

$$½\,(1 + 1/t^2)\, e^{x/2\,(t - 1/t)} = \Sigma_n n t^{n-1} J_n(x)$$

Replacing the expression $[e^{x/2\,(t - 1/t)}]$ on the LHS again by $\Sigma_n t^n J_n(x)$:

$$(x/2)\,(1 + 1/t^2)\, \Sigma_n t^n J_n(x) = \Sigma_n n t^{n-1} J_n(x)$$

$$\Rightarrow (x/2)\, \Sigma_n [t^n J_n(x) + t^{n-2} J_n(x)] = \Sigma_n n t^{n-1} J_n(x)$$

Equating the coefficient of $\mathbf{t^{m-1}}$ on both sides :

$$(x/2)\,[J_{m-1}(x) + J_{m+1}(x)] = m\, J_m(x)$$

$$\Rightarrow [J_{m-1}(x) + J_{m+1}(x)] = (2m/x)\, J_m(x) \quad \text{---- (2)}$$

One can also generate the following relations from (1) and (2) :

$$\frac{d}{dx}\{x^n J_n(x)\} = n x^{n-1} J_n(x) + x^n J_n'(x)$$
$$= x^n [(n/x) J_n(x) + J_n'(x)]$$
$$= x^n [\, \{J_{n-1}(x) + J_{n+1}(x)\}/2 + \{J_{n-1}(x) - J_{n+1}(x)\}/2 \,]$$
$$= x^n J_{n-1}(x) \quad \text{---- (3)}$$

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -n x^{-n-1} J_n(x) + x^{-n} J'_n(x)$$
$$= x^{-n} [-(n/x) J_n(x) + J'_n(x)]$$
$$= x^{-n} [-\{J_{n-1}(x) + J_{n+1}(x)\}/2 + \{J_{n-1}(x) - J_{n+1}(x)\}/2 \,]$$
$$= -x^{-n} J_{n+1}(x) \quad \text{---- (4)}$$