Forced Vibrations

Forced vibrations and resonance:

The study of forced vibrations and resonance is of special interest in sound, as sound is detected by forced vibration or resonance it produces in the receiver. Resonance is special kind of forced vibration.

Forced vibrations:

Since the vibrating system gradually loses its amplitude, energy has to be supplied from without if the system is to be maintained in vibration. If an external periodic force acts on a vibrating system, the system tends to vibrate with its own natural frequency, while the applied (driving) force tries to impress its own frequency of vibration on the system. Initially, vibrations of both frequencies are simultaneously present. In course of time, the natural vibration dies away due to the resisting forces that operate. Finally, the vibrations due to the impressed driving force are fully established. The vibration of a system with period the same as that of an impressed periodic force is called forced vibration.

Resonance:

The amplitude of forced vibration is generally small. If the period of applied force agrees with the natural period of the vibrating system the vibrations build up quickly and grow to relatively large amplitude even if the impressed force is small. As the motion grows, the resistance to the motion also increases. A state of steady oscillation is reached when the energy supplied by the external source is fully used up in overcoming the resistance to the motion. The particular case of forced vibration where the applied force agrees in period with natural period of unresisted vibration of the system, communicating maximum energy to the system, is known as resonance. A resonant vibrator is also termed as sympathetic vibrator.

Theoretically, there are two cases of resonance to distinguish, (i) amplitude resonance, and (ii) velocity or energy resonance.

Amplitude resonance:

In this case the amplitude of the forced vibration acquires a maximum value. Since the potential energy is proportional to the square of amplitude, the system will have maximum potential energy under this condition.

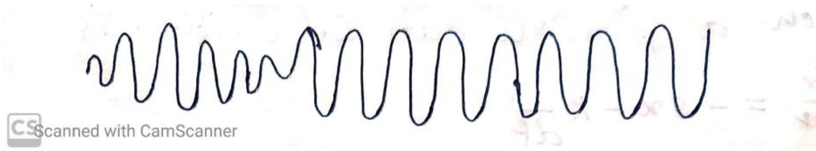
Velocity resonance:

In this case the velocity is a maximum. This corresponds to the maximum energy transfer between the forcing system and the forced system, and gives the system maximum kinetic energy.

When the damping is negligible, both the resonances occur at practically the same frequency, which is the natural frequency of the system. When there is appreciable damping, the two resonances occur at slightly different frequencies. Velocity resonance occurs at the frequency equal to the undamped frequency of the system in forced vibration; this is more important case.

Characteristics of forced vibration:

(i) Initially, vibrations with the natural frequency of the vibrating system and the frequency of the impressed force are both present simultaneously. If the frequencies are close enough they may form beats. In course of time the natural vibration dies out and the system vibrates with constant amplitude at the driving force. This is known as steady state vibration.



(ii) The amplitude of forced vibration is generally small except in case of resonance, when its amplitude may be quite large.

Difference between free and forced vibrations:

(i) Free vibration is executed by a system under the action of its own elastic forces without being subjected to any external force. But, forced vibration is executed by a system under the action of an externally applied periodic force.

(ii) The initial amplitude of free vibration may have any value, large or small, depending on the initial supply of energy. With damping the amplitude diminishes exponentially with time. Amplitude of forced vibration is generally small except when resonance occurs. Under resonance conditions the amplitude of vibrations is large. Near resonance the amplitude increases rapidly as the frequency of the applied force approaches the frequency of the unresisted free vibration.

(iii) Frequency of free vibration depends on the mass and elasticity which are treated as localized. Frequency of forced vibration in steady state is equal to that of the applied periodic force.

(iv) Free vibration eventually ceases due the action of resisting forces. But, forced vibration continues as long as the applied force acts.

Forced vibration:

A particle executing damped simple harmonic motion is subjected to two forces: (i) restoring force proportional to the distance, (ii) a resisting or retarding force proportional to the instantaneous velocity of the particle. Therefore, the particle executing damped SHM loses energy and hence its amplitude decreases due to damping. If the particle is to execute oscillatory motion without losing amplitude, a force of constant amplitude of the form $FCospt$ or $FSinpt$ (F is the amplitude of the force and p is its circular frequency) has to be applied to the vibrating particle.

If ‘m’ be the mass of the vibrating particle, ‘K’ the damping constant, ‘s’ is the stiffness/spring constant, and $FCospt$ is the externally applied periodic force then the equation of motion of the forced vibrating particle can be written in the following form

$$m\ddot{x} = -sx - K\dot{x} + FCospt$$

$$\therefore m\ddot{x} + K\dot{x} + sx = FCospt$$

This is a non-homogeneous second order differential equation. There are several ways of solving this differential equation. We shall find it convenient to use complex numbers.

To get the equation of motion in complex form, let x_1 and x_2 be the displacements when the forces are represented by $FCospt$ and $FSinpt$ respectively.

$$\therefore m\ddot{x}_1 + K\dot{x}_1 + sx_1 = FCospt$$

and $m\ddot{x}_2 + K\dot{x}_2 + sx_2 = FSinpt$

$$\therefore \ddot{x}_1 + 2b\dot{x}_1 + \omega^2 x_1 = fCospt$$

and $\ddot{x}_2 + 2b\dot{x}_2 + \omega^2 x_2 = fSinpt$ ....(1)

where $2b = \frac{K}{m}$, $\omega^2 = \frac{s}{m}$ and $\mathrm{f} = \frac{F}{m}$.

Now, from eq. (1) we can write,

$(\ddot{x}_1 + j\ddot{x}_2) + 2\,\mathrm{b}(\dot{x}_1 + j\dot{x}_2) + \omega^2(x_1 + jx_2) = f(Cospt + jSinpt)$ ....(2)

where $j = \sqrt{-1}$.

If we write $X = x_1 + jx_2$, where X is a complex quantity, the above equation reduces to

$\ddot{X} + 2b\dot{X} + \omega^2 X = fe^{jpt}$ ....(3)

Complementary function of eq. (3) can be found from the equation

$\ddot{X} + 2b\dot{X} + \omega^2 X = 0$ ....(4)

When b < ω (which is usually the case) the solution of eq. (4) is

$$\begin{aligned} X &= e^{-\mathrm{bt}}(A_1 Cos\sqrt{w^2 - b^2}\ \mathrm{t} + A_2 Sin\sqrt{w^2 - b^2}\ \mathrm{t}) \\ &= ae^{-\mathrm{bt}} Cos(\sqrt{w^2 - b^2}\ \mathrm{t} - \varepsilon) \end{aligned}$$ ....(5)

where $A_1 = a\,\mathrm{Cos}\,\varepsilon$ and $A_2 = aSin\varepsilon$.

To find the particular solution of eq. (3) we put $X = Ae^{jpt}$ ....(6) in eq. (3) where A is a complex quantity.

$$\therefore (-p^2 + 2b.jp + w^2)Ae^{jpt} = fe^{jpt}$$

$\therefore A = \frac{f}{(w^2 - p^2) + j.2bp} = \frac{(w^2 - p^2) - j.2bp}{(w^2 - p^2)^2 + 4b^2p^2} f$ ....(7)

Let $w^2 - p^2 = CCos\phi$ and $2bp = CSin\phi$.

$$\therefore C = \sqrt{(w^2 - p^2)^2 + 4b^2p^2} \text{ and } \tan\phi = \frac{2bp}{(w^2 - p^2)} \text{ ....(8)}$$

$$\therefore \begin{aligned} A &= \frac{Ce^{-j\phi}}{C^2} f = \frac{f}{C} e^{-j\phi} \\ &= \frac{f}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} e^{-j\phi} \end{aligned} \text{ ....(9)}$$

$$\therefore X = \frac{f}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} e^{j(pt-\phi)} \text{ ....(10)}$$

$$\therefore x_1 + jx_2 = \frac{f}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} \{\mathrm{Cos}(pt-\phi) + jSin(pt-\phi)\} \text{ ....(11)}$$

$$\therefore x_1 = \frac{f}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} \mathrm{Cos}(pt-\phi)$$

and $x_2 = \frac{f}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} \mathrm{Sin}(pt-\phi)$ ....(12)

x_1 and x_2 represent the particular solutions when the applied force is represented by $FCospt$ and $FSinpt$ respectively.

$$\text{Now, } \begin{aligned} &\frac{f}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} = \frac{F/m}{\sqrt{(w^2 - p^2)^2 + 4b^2p^2}} = \frac{F}{\sqrt{(\mathrm{m}\,w^2 - mp^2)^2 + 4b^2p^2m^2}} \\ &= \frac{F}{\sqrt{(\mathrm{s} - mp^2)^2 + K^2p^2}} = \frac{F}{p\sqrt{(\frac{s}{p} - mp)^2 + K^2}} = \frac{F}{pZ_m} \end{aligned} \text{ ....(13)}$$

where $Z_m = \sqrt{(\frac{s}{p} - mp)^2 + K^2}$ ....(14) is known as the mechanical impedance and

$\chi_m = \mathrm{mp} - \frac{s}{p}$ is known as the mechanical reactance.

$$\therefore \begin{aligned} x_1 &= \frac{F}{pZ_m} \mathrm{Cos}(pt-\phi) \\ x_2 &= \frac{F}{pZ_m} \mathrm{Sin}(pt-\phi) \end{aligned} \text{ ....(15)}$$

If we write the differential equation of forced vibration as

$$\ddot{X} + 2b\dot{X} + \omega^2 X = fe^{jpt}$$

Taking the real part of the solution as in eq. (15) and following eq.(5), the total solution is given by

$$x = ae^{-\mathrm{bt}} Cos(\sqrt{w^2 - b^2}\ \mathrm{t} - \varepsilon) + \frac{F}{pZ_m} \mathrm{Cos}(pt - \phi) \text{ ....(16)}$$

The first term in eq. (16) gives the natural vibration of the system i.e. motion it executes when disturbed from the equilibrium and left to itself.

The second term represents the forced vibration i.e. the response of the particle to the externally imposed force, called the driving force, which maintains the vibration. At the initial stage, effect of both the term is present; but with time the natural motion dies down due to the factor $e^{-\mathrm{bt}}$ and we are left with steady state motion represented by the second term. If ω and p are close together in value, the two vibrations produce beats, which last longer if b is smaller.

The motion represented by the first term in eq. (16) is called the transient. Transient appears both when the driving force is applied and when it is removed.

When b is greater than or equal to ω, the corresponding solution of $m\ddot{x} + K\dot{x} + sx = 0$ will give the transient. Such case is however is of little practical interest.

Hence the steady state solution is given by

$$x = \frac{F}{pZ_m} \mathrm{Cos}(pt - \phi) \text{ ....(17)}$$

Velocity at steady state:

$$\mathrm{v} = \frac{dx}{dt} = -\frac{F}{pZ_m}.\mathrm{pSin}(pt - \phi)$$

$$= -\frac{F}{Z_m} \mathrm{Sin}(pt - \phi)$$

$$= \frac{F}{Z_m} Cos\{(pt - \phi) + \frac{\pi}{2}\}$$

$$= \frac{F}{Z_m} Cos(pt - \theta) \text{..................(18)}$$

where $\theta = \phi - \frac{\pi}{2}$ ....(19)

Now, $\phi = \theta + \frac{\pi}{2}$

$$\therefore \tan\phi = \tan(\theta + \frac{\pi}{2}) = \frac{2bp}{(w^2 - p^2)}$$

$$\therefore -\cot\theta = \frac{2bp}{(w^2 - p^2)}$$

$$\therefore \tan\theta = \frac{(p^2 - w^2)}{2bp} \text{......................(20)}$$

Here θ is the phase lag of velocity behind the driving force.

Resonance:

In forced vibration the displacement and velocity amplitudes of the driven system depend on the frequency of the driving force. The value of p when the value of amplitude of the driven system becomes maximum can be calculated and we say resonance occurs between the driver and the driven system. Two cases may be distinguished, (i) amplitude resonance and (ii) velocity resonance.

(i) Amplitude resonance:

The displacement amplitude is given by

$$A = \frac{F}{pZ_m} = \frac{F}{\sqrt{(\mathrm{s} - mp^2)^2 + K^2 p^2}}$$

If A is to be the maximum for some value of p, the denominator $\sqrt{(\mathrm{s} - mp^2)^2 + K^2 p^2}$ will be a minimum at that value i.e. $(\mathrm{s} - mp^2)^2 + K^2 p^2$ will also be a minimum. Hence at this value of p we have,

$$\frac{d}{dp}\{(\mathrm{s} - mp^2)^2 + K^2 p^2\} = 0$$

Or, $2(\mathrm{s} - mp^2)(-2\,\mathrm{mp}) + K^2.2p = 0$

Or, $(\mathrm{s} - mp^2).2mp = 2K^2 p$

Or, $\mathrm{s} - mp^2 = \frac{K^2}{2m}$

Or, $mp^2 = s - \frac{K^2}{2m}$

$$\therefore p^2 = \frac{s}{m} - \frac{K^2}{2m^2}$$

Now, $\frac{s}{m} = \omega^2$, where ω is the natural frequency of the driven system without damping and $\frac{K}{m} = 2b$, where b is the damping constant. Using these values we have,

$p^2 = \omega^2 - 2b^2$

$\therefore\ p = \sqrt{\omega^2 - 2b^2}$

This is the frequency p of the driving force at which the amplitude resonance occur.

$$A_{\max} = \frac{F}{\sqrt{(\mathrm{s} - mp^2)^2 + K^2 p^2}} = \frac{F}{\sqrt{\{\mathrm{m}\,\omega^2 - m(\omega^2 - 2b^2)\}^2 + K^2(\omega^2 - 2b^2)}}$$

$$= \frac{F}{\sqrt{(\mathrm{m}\,\omega^2 - m\omega^2 + 2b^2\,\mathrm{m})^2 + 4b^2m^2.(\omega^2 - 2b^2)}}$$

$$= \frac{F}{\sqrt{4b^4\,\mathrm{m}^2 + 4b^2m^2\omega^2 - 8b^4m^2}} = \frac{F}{\sqrt{4b^2m^2\omega^2 - 4b^4m^2}} = \frac{F}{\sqrt{4b^2m^2(\omega^2 - b^2)}}$$

$$= \frac{F}{2bm\sqrt{\omega^2 - b^2}}$$

This is the value of the displacement amplitude at amplitude resonance.

Since the potential energy is given by $\frac{1}{2}sx^2$, its value becomes a maximum (with varying p) when the amplitude is a maximum. Thus the amplitude resonance corresponds to the maximum potential energy of the driven system as the frequency p of the driver is varied (keeping the amplitude F of the driving force constant).

Note: The frequency $p = \sqrt{\omega^2 - 2b^2}$ for amplitude resonance is neither equal to the natural frequency ω of the driven system nor the frequency $\sqrt{\omega^2 - 2b^2}$ of the damped vibration. The resonance frequency is lower than both.

(ii) Velocity resonance:

The velocity amplitude is given by

$$V = \frac{F}{Z_m} = \frac{F}{\sqrt{(\mathrm{mp} - \frac{s}{p})^2 + K^2}}$$

With p as the only variable, V will be maximum when the denominator $\sqrt{(\mathrm{mp} - \frac{s}{p})^2 + K^2}$ will be minimum and $\sqrt{(\mathrm{mp} - \frac{s}{p})^2 + K^2}$ is minimum when $\mathrm{mp} - \frac{s}{p} = 0$ i.e. when mechanical reactance χ_m will vanish.

$\therefore\ \mathrm{mp} - \frac{s}{p} = 0$

Or, $p^2 = \frac{s}{m} = \omega^2$

$\therefore p = \omega$

The velocity resonance frequency is thus equal to the natural frequency of the driven system without damping.

$\therefore V_{\max} = \frac{F}{K}$

Since the kinetic energy is given by $\frac{1}{2}mv^2 = \frac{1}{2}m(\frac{dx}{dt})^2$, its value reaches maximum (with varying p) when the velocity amplitude reaches maximum. Thus kinetic energy of the driven system is a maximum at velocity resonance.

$$(KE)_{res} = \frac{1}{2}mV_{res}{}^2 = \frac{mF^2}{2K^2}$$

When b << ω i.e. for small damping the condition for the two resonances become practically identical.

Note: The condition for velocity resonance ($\text{mp} - \frac{s}{p} = 0$) is that of disappearance of mechanical reactance. Thus we may say that resonance occurs at that frequency of the driven force at which the mechanical reactance vanishes. The mechanical impedance at resonance is

$$(Z_m)_{\text{res}} = \text{K}$$

Power in forced vibration and resonance:

The instantaneous rate of work done by the driving force in steady state is given by

$$P(t) = \text{FCos}\, pt.\frac{dx}{dt} = \text{FCos}\, pt.\frac{F}{Z_m}Cos(pt-\theta)$$

$$= \frac{F^2}{Z_m}(\text{Cos}^2\, ptCos\theta + SinptCosptSin\theta)$$

$$= \frac{F^2}{Z_m}(\text{Cos}^2\, ptCos\theta + \frac{1}{2}Sin2ptSin\theta)$$

The average value of power is

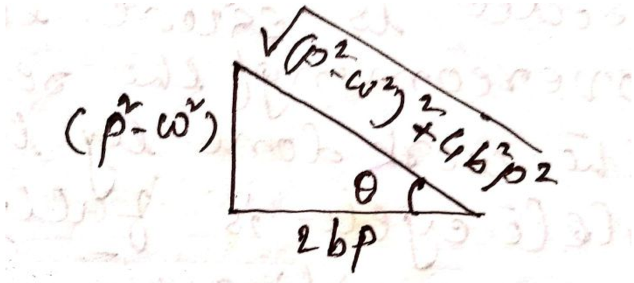
$$< \mathrm{P} >= \frac{F^2}{Z_m}.\frac{1}{T}\int_0^T (\mathrm{Cos}^2\, pt Cos\theta + \frac{1}{2} Sin2pt Sin\theta)dt$$

$$= \frac{F^2}{Z_m}.\frac{1}{T}(Cos\theta \int_0^T \mathrm{Cos}^2\, pt\mathrm{dt} + \frac{1}{2} Sin\theta \int_0^T Sin2pt\mathrm{dt})$$

$$= \frac{F^2}{Z_m}.\frac{1}{T}.Cos\theta.\frac{T}{2}$$

[$\because \int_0^T \mathrm{Cos}^2\, pt\mathrm{dt} = \frac{T}{2}$ and $\int_0^T Sin2pt\mathrm{dt} = 0$]

$$\therefore < \mathrm{P} >= \frac{F^2}{2Z_m} Cos\theta = \frac{F^2}{2Z_m} \frac{2bp}{\sqrt{(\frac{s}{p} - \mathrm{mp})^2 + K^2}}$$



Now, $Z_m = \sqrt{(\frac{s}{p} - \mathrm{mp})^2 + K^2} = \frac{m}{p}\sqrt{(w^2 - p^2)^2 + 4b^2p^2}$

$$\therefore < \mathrm{P} >= \frac{F^2}{2Z_m}.\frac{\frac{K}{m}p}{\frac{pZ_m}{m}} = \frac{F^2K}{2Z_m{}^2}$$

Since at resonance $Z_m = K$,

$$< \mathrm{P} >_{res} = \frac{F^2K}{2K^2} = \frac{F^2}{2K}$$

This is the maximum value of power that can be obtained by varying p alone.

Work done against the retarding force:

The retarding force acting on the particle is $K\dot{x} = Kv$.

The instantaneous rate of work done against the retarding force is

$$\mathrm{W}(t) = K\dot{x}.\frac{dx}{dt} = K\dot{x}^2 = K.\frac{F^2}{Z_m{}^2}Cos^2(pt - \theta)$$

Hence the average power spent in overcoming the retarding force is

$$< \mathrm{W} >= \frac{F^2 K}{Z_m{}^2}.\frac{1}{T}\int_0^T Cos^2(pt-\theta)dt$$

$$= \frac{F^2 K}{Z_m{}^2}.\frac{1}{T}.\frac{T}{2} = \frac{F^2 K}{2Z_m{}^2}$$

Conclusion:

Thus the average instantaneous rate of work done by the driving force in steady state is equal to the power spent in overcoming the resistance. In steady state the work done by the driving force in a complete cycle is fully spent in overcoming the force resisting the motion. This also holds at resonance.

Hence an oscillatory body like pendulum under this condition will oscillate indefinitely.

Power factor:

$$< \mathrm{P} >= \frac{F^2}{2Z_m} Cos\theta = \frac{F}{\sqrt{2}}.\frac{F}{\sqrt{2}Z_m} Cos\theta$$

= RMS force X RMS velocity X $Cos\theta$

Because of the phase difference θ between the applied force and velocity of the particle, the power is not equal to the product of their effective value, but to this quantity multiplied by $Cos\theta$. $Cos\theta$ is hence called the power factor.

$$Cos\theta = \frac{2bp}{\sqrt{(w^2-p^2)^2+4b^2p^2}} = \frac{2bp}{\frac{pZ_m}{m}} = \frac{2bm}{Z_m} = \frac{K}{Z_m}$$

Since at resonance $Z_m = K$, $Cos\theta = 1$.

$\therefore \theta = 0$

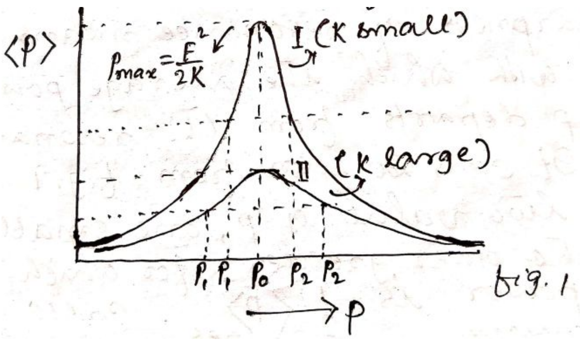
Hence at resonance the applied force and velocity are in phase.

Sharpness of resonance:

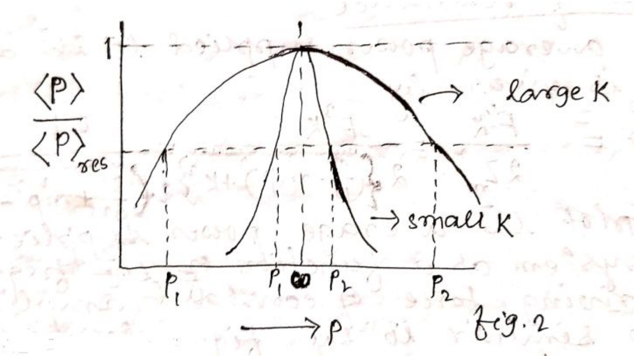
The average power supplied to the driven system is given by

$$< \mathrm{P} >= \frac{F^2 K}{2Z_m{}^2} = \frac{F^2 K}{2\{(\frac{s}{p} - \mathrm{m}\, p)^2 + K^2\}} = \frac{F^2 K}{2(\frac{s^2}{p^2} + \mathrm{m}^2 p^2 - 2ms + K^2)}$$

If we plot the average power supplied to the driven system as a function of the frequency p of the driving force of constant amplitude, a curve similar to the figure below is obtained.



It has a maximum value $\frac{F^2}{2K}$ at the resonance frequency $p_0 = \omega$ and falls off more or less rapidly at lower and higher frequencies. When K is small (curve I), the curve acquires a high peak value. When K is large (curve II), the peak is low. For small values of K, the value of the average power $< \mathrm{P} >$ falls off rapidly as the frequency p of the driving force differs more and more from $p_0 = \omega$. Resonance in such a case is said to be sharp. On the other hand, when K is large $< \mathrm{P} >$ changes much more slowly as p departs from ω. Resonance in such a case is said to be broad or flat. The differences in sharpness of resonance due to differences in the value of K are very clearly seen when the ratio $\frac{< \mathrm{P} >}{< \mathrm{P} >_{res}}$ is plotted against p as shown in the figure below.



The sharpness of resonance means the rapidity with which the average power diminishes as p departs from the resonance frequency ω. It can be seen from fig.1 that there are two values of p, one smaller than the resonance frequency ω and other greater than ω, for which the average power is $\frac{1}{2} < \mathrm{P} >_{res}$. These two frequencies are known as half-power frequencies. Frequency p_1 which is lower than ω is called lower half-power frequency and frequency p_2 which is greater than ω is known as the upper half-power frequency. $p_2 - p_1$ is called the band-width. The quality factor Q is defined as the ratio of resonance frequency and band-with.

$$Q = \frac{\omega}{p_2 - p_1}$$

Quality factor can be taken as the measure of sharpness of resonance.

We have, $<\mathrm{P}>_{res}=\dfrac{F^2}{2K}$ while $<\mathrm{P}>=\dfrac{F^2K}{2{Z_m}^2}$. The values of p_1 and p_2 can be found by solving the relations $<\mathrm{P}>=\dfrac{1}{2}<\mathrm{P}>_{res}=\dfrac{F^2}{4K}$ and $<\mathrm{P}>=\dfrac{F^2K}{2{Z_m}^2}$.

$$\therefore \frac{F^2K}{2{Z_m}^2}=\frac{F^2}{4K}$$

Or, ${Z_m}^2=2K^2$

Or, $K^2+\chi^2=2K^2$

Or, $\chi^2=K^2$

$\therefore \chi=\pm K$

Now, $\chi=mp-\dfrac{s}{p}$.

$\therefore mp_1-\dfrac{s}{p_1}=-K$ and $mp_2-\dfrac{s}{p_2}=K$

From the first equation we have,

$$s=(K+mp_1)p_1$$

Now putting the value of s in the second equation we have,

$$mp_2-\frac{p_1}{p_2}(K+mp_1)=K$$

Or, $m(p_2-\dfrac{{p_1}^2}{p_2})=\mathrm{K}(1+\dfrac{p_1}{p_2})$

Or, $m.\dfrac{{p_2}^2-{p_1}^2}{p_2}=K.\dfrac{p_2+p_1}{p_2}$

Or, $m(p_2-p_1)=K$

$\therefore p_2-p_1=\dfrac{K}{m}=2b$

This gives $Q=\dfrac{\omega}{p_2-p_1}=\dfrac{m\omega}{K}=\dfrac{\omega}{2b}$.

Thus sharp resonance is associated with high resonance frequency and small damping.

Problems:

1. Show that for forced vibration the total energy of the vibrating system is not constant, and also prove that in such a case

$$\frac{average\ PE}{average\ KE} = \frac{\omega_0{}^2}{\omega^2}$$

where $\omega_0 = \sqrt{\frac{s}{m}}$ **.**

Sol:

For a particle executing forced vibration displacement is given by

$x(t) = ACos(\omega t - \phi)$,

where A is the amplitude of vibration.

The kinetic energy of a particle of mass m in forced vibration is given by

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2 Sin^2(\omega t - \phi)$$

The potential energy of the particle is given by

$$PE = \frac{1}{2}sx^2 = \frac{1}{2}sA^2Cos^2(\omega t - \phi) = \frac{1}{2}m\omega_0{}^2 A^2Cos^2(\omega t - \phi)$$

Hence the total energy is given by

$$E = KE + PE = \frac{1}{2}m\omega^2 A^2 Sin^2(\omega t - \phi) + \frac{1}{2}m\omega_0{}^2 A^2 Cos^2(\omega t - \phi)$$

This shows that E is a function of time t, and therefore is not a constant.

Now, the average Kinetic energy over a complete cycle is

$$< KE >= \frac{1}{2}m\omega^2 A^2.\frac{1}{T}\int_0^T Sin^2(\omega t - \phi)dt = \frac{1}{2}m\omega^2 A^2.\frac{1}{T}.\frac{T}{2} = \frac{1}{4}m\omega^2 A^2$$

Again, the average potential energy over a complete cycle is given by

$$< PE >= \frac{1}{2}m\omega_0{}^2 A^2.\frac{1}{T}\int_0^T Cos^2(\omega t - \phi)dt = \frac{1}{2}m\omega_0{}^2 A^2.\frac{1}{T}.\frac{T}{2} = \frac{1}{4}m\omega_0{}^2 A^2$$

$$\therefore \frac{average\ PE}{average\ KE} = \frac{<KE>}{<PE>} = \frac{\frac{1}{4} m\omega_0^2 A^2}{\frac{1}{4} m\omega^2 A^2} = \frac{\omega_0^2}{\omega^2}$$

2. If ω_1 and ω_2 be the half-power frequencies, and ω_0 is the resonant frequency of a forced vibrating particle, show that $\omega_0^2 = \omega_1 \omega_2$.

Sol:

For a forced vibrating system the average power is given by

$$<P> = \frac{F^2 K}{2 Z_m^2}$$

where the mechanical impedance $Z_m = \sqrt{(\frac{s}{\omega} - m\omega)^2 + K^2}$.

$<P>$ becomes maximum at resonance when $\omega = \omega_0$ and is given by

$$<\mathrm{P}>_{res} = \frac{F^2}{2K}$$

Half-power frequencies are frequencies for which $<P> = \frac{1}{2} <\mathrm{P}>_{res}$.

$$\therefore \frac{F^2 K}{2 Z_m^2} = \frac{1}{2} \frac{F^2}{2K}$$

Or, $Z_m^2 = 2K^2$

Or, $(\frac{s}{\omega} - m\omega)^2 + K^2 = 2K^2$

Or, $(\frac{s}{\omega} - m\omega)^2 = K^2$

Or, $(\frac{m\omega_0^2}{\omega} - m\omega)^2 = K^2$

Or, $\frac{m^2}{\omega^2}(\omega_0^2 - \omega^2)^2 = K^2$

Or, $(\omega_0{}^2 - \omega^2)^2 = K^2.\dfrac{\omega^2}{m^2} = 4b^2m^2.\dfrac{\omega^2}{m^2} = 4b^2\omega^2$

Or, $\omega_0{}^4 + \omega^4 - 2\omega_0{}^2\omega^2 = 4b^2\omega^2$

Or, $\omega^4 - 2(\omega_0{}^2 + 2b^2)\omega^2 + \omega_0{}^4 = 0$

This is a quadratic equation in ω^2 which has two roots say $\omega_1{}^2$ and $\omega_2{}^2$.

$\therefore \omega_1{}^2\omega_2{}^2 = \omega_0{}^4$

$\therefore \omega_0{}^2 = \omega_1\omega_2$