Legendre Equation

The following differential equation is known as the Legendre Differential Equation, after the **French mathematician Adrien-Marie Legendre**, who discovered them in 1782.

$$\mathbf{(1 - x^2)\, d^2y/dx^2 - 2x\, dy/dx + \lambda\, y = 0} \qquad \text{----- (1)}$$

Dividing by $(1 - x^2)$, the equation may be re-written as :

$$d^2y/dx^2 - \{2x/(1 - x^2)\}\, dy/dx + \{\lambda/(1 - x^2)\}\, y = 0$$

The dy/dx term and the y-term have simple poles at $\mathbf{x = \pm 1}$, hence those are the **regular singular points** of this differential equation.

However, we wish to solve this differential equation about $x = 0$, which is an ordinary point. So, the simple power series method would have been sufficient, but we shall adopt the Frobenius' method (named after the **German mathematician Ferdinand Georg Frobenius**).

According to Frobenius' Theorem, we can expect a solution of the form :

$$y = \Sigma\, C_n x^{n+s} \Rightarrow dy/dx = \Sigma\, (n + s)\, C_n x^{n+s-1}$$

$$\Rightarrow d^2y/dx^2 = \Sigma\, (n + s)\,(n + s - 1)\, C_n x^{n+s-2}$$

Substituting in the differential equation :

$$(1 - x^2)\, \Sigma\, (n + s)\,(n + s - 1)\, C_n x^{n+s-2} - 2\, \Sigma\, (n + s)\, C_n x^{n+s} + \lambda\, \Sigma\, C_n x^{n+s} = 0 \quad \text{----- (2)}$$

Since this is an identity, the coefficient of each power of x must separately vanish.

Equating the coefficient of the lowest power of x, i.e., $x^{s-2} \to 0$, we have :

$$s\,(s - 1)\, C_0 = 0 \qquad \text{----- (3)}$$

[Contribution only comes from the first term with n = 0]

$$\Rightarrow s = 0, \text{ or, } s = 1, \text{ or, } C_0 = 0$$

Similarly, equating the coefficient of $x^{s-1} \to 0$, we have :

$$(s + 1)\, s\, C_1 = 0 \qquad \text{----- (4)}$$

[Contribution again comes only from the first term with n = 1]

$$\Rightarrow s = -1 , \text{ or, } s = 0, \text{ or, } C_1 = 0$$

If we wish to keep both C_0 and C_1 non-zero (we shall require two constants to construct the general solution), we are only left with the choice : $s = 0$.

For s = 0,

our eqn. (2) reduces to :

$$(1 - x^2)\, \Sigma\, n\,(n - 1)\, C_n x^{n-2} - 2\, \Sigma\, n\, C_n x^n + \lambda\, \Sigma\, C_n x^n = 0$$

$$\Rightarrow \Sigma\, n\,(n - 1)\, C_n x^{n-2} = \Sigma\, n\,(n - 1)\, C_n x^n + 2\, \Sigma\, n\, C_n x^n - \lambda\, \Sigma\, C_n x^n$$

$$= \Sigma\, [n\,(n - 1) + 2n - \lambda]\, C_n x^n$$

$$= \Sigma\, [n\,(n + 1) - \lambda]\, C_n x^n$$

Equating the coefficient of $x^m \to 0$, we have :

$$(m + 2)\,(m + 1)\, C_{m+2} = [m\,(m + 1) - \lambda]\, C_m$$

$$\Rightarrow C_{m+2} = C_m\, [m\,(m + 1) - \lambda] / (m + 1)\,(m + 2) \qquad \text{----- (5)}$$

Eqn. (5) is known as the **recursion relation**, which allows us to express C_2, C_4, etc., in terms of C_0 and C_3, C_5, etc., in terms of C_1. Thus, we obtain two series constituted of the even and the odd power terms. For example,

$C_2 = C_0 [-\lambda] / 2!$, $C_4 = C_2 [2\times3 - \lambda] / (3\times4) = C_0 [(-\lambda)(6-\lambda)] / 4!$, etc.

$C_3 = C_1 [1\times2 - \lambda] / 3!$, $C_5 = C_3 [3\times4 - \lambda] / (4\times5) = C_1 [(2-\lambda)(12-\lambda)] / 5!$, etc.

Thus, $y = C_0 [1 + (-\lambda/2!)\, x^2 + \{(-\lambda)(6-\lambda)/4!\}\, x^4 + \ldots]$

$+ C_1 [x + \{(2-\lambda)/3!\}\, x^3 + \{(2-\lambda)(12-\lambda)/5!\}\, x^5 + \ldots]$. ----- (6)

Note that, the constants C_0 and C_1 remains undetermined. They play the role of the two arbitrary constants, which are necessary for constructing the general solution.

It can be shown, that the series thus generated, converges for $|x| < 1$ but **not** for $|x| \geq 1$. Now, Legendre Equation usually appears in Physics, from the partial differential equations involving the Laplacian operator ∇^2, in spherical polar coordinates. One obtains eqn.(1) by substituting : $\boldsymbol{\cos\theta = x}$. That means, we need not bother about the values $x > 1$, but the values $\boldsymbol{x = \pm 1}$ **i.e.,** $\boldsymbol{\theta = 0, \pi}$ are relevant where finite solutions are necessary.

The only escape route is to make the series terminate, which is possible if C_{n+2} (and the subsequent terms) vanishes for some values of n. This is only possible if λ is of the form : $\boldsymbol{\ell(\ell+1)}$**, for some positive integer '$\boldsymbol{\ell}$'**. This reduces the series into a ℓ-th order polynomial, known as the **Legendre Polynomial**.

One should note that the even series terminates if ℓ is even and the odd one if ℓ is odd. So both the series cannot be truncated into polynomials. The one which remains an infinite series, still suffers from the convergence problem, hence its coefficient is chosen to vanish for maintaining the boundary condition : y is finite for $x = \pm 1$.

[For example, if $y = Ae^{x} + Be^{-x}$ and we require that y is finite for $x \to \infty$, then we must choose : $A = 0$.]

Another approach :

We quickly recapitulate what we learnt about Legendre Differential Equation. The following differential equation is known as the Legendre Differential Equation, after the **French mathematician Adrien-Marie Legendre**.

$$\boldsymbol{(1-x^2)\, d^2y/dx^2 - 2x\, dy/dx + \lambda\, y = 0} \qquad \text{----- (1)}$$

We tried to solve it by applying Frobenius' method (named after the **German mathematician Ferdinand Georg Frobenius**) and looked for a solution of the form : $\boldsymbol{y = \Sigma\, C_n\, x^{n+s}}$

This led to : $dy/dx = \Sigma\, (n+s)\, C_n\, x^{n+s-1}$

$$\Rightarrow\ d^2y/dx^2 = \Sigma\, (n+s)(n+s-1)\, C_n\, x^{n+s-2}$$

Substituting in the differential equation, we obtained :

$$(1-x^2)\, \Sigma\, (n+s)(n+s-1)\, C_n\, x^{n+s-2} - 2\, \Sigma\, (n+s)\, C_n\, x^{n+s} + \lambda\, \Sigma\, C_n\, x^{n+s} = 0 \quad \text{----- (2)}$$

$\Rightarrow \Sigma\, (n+s)(n+s-1)\, C_n\, x^{n+s-2}$

$$= \Sigma\, (n+s)(n+s-1)\, C_n\, x^{n+s} + 2\, \Sigma\, (n+s)\, C_n\, x^{n+s} - \lambda\, \Sigma\, C_n\, x^{n+s}$$

$$= \Sigma\, (n+s)(n+s+1)\, C_n\, x^{n+s} - \lambda\, \Sigma\, C_n\, x^{n+s}$$

Since this is an identity, the coefficient of each power of x must separately vanish.

Equating the coefficient of the lowest power of x, i.e., $x^{s-2} \to 0$, we had :

$$s\,(s-1)\, C_0 = 0$$

$$\Rightarrow\ s = 0,\ \text{or},\ s = 1,\ \text{or},\ C_0 = 0$$

Similarly, equating the coefficient of $x^{s-1} \to 0$, we had :

$$(s + 1)\, s\ C_1 = 0$$

$$\Rightarrow\ s = -1\ ,\ \text{or},\ s = 0,\ \text{or},\ C_1 = 0$$

We wished to keep both C_0 and C_1 non-zero, hence, we are only left with the choice : **$s = 0$**.
This provided us a **recursion relation**, which relates C_2, C_4, etc., with C_0 and C_3, C_5, etc., with C_1, thus leading to a general solution with two arbitrary constants C_0 and C_1.

If however, we set $C_1 = 0$ **by choice**, C_3, C_5, etc., will all vanish and we shall only get the even-power series. To get another solution, which is necessary for constructing the general solution.

Equating the coefficient of coefficient of x^{m+s} on both sides of (2), we have :

$$(m + s + 2)(m + s + 1)\, C_{m+2} = [(m + s)(m + s + 1) - \lambda]\, C_m$$

If we choose **$s = 1$**, this reduces to :

$$C_{m+2} = C_m\, [(m + 1)(m + 2) - \lambda] / (m + 2)(m + 3) \qquad \text{----- (3)}$$

The choice $s = 1$ compels us to choose $C_1 = 0$ and hence $C_3, C_5, \cdots = 0$.
Now,

$$C_2 = C_0\, [2 - \lambda] / 3!\ ,\ C_4 = C_2\, [3\times 4 - \lambda] / (4\times 5) = C_0\, [(2-\lambda)(12 - \lambda)] / 5!,\ \text{etc.}$$

Note that the relation is identical to the one obtained for the **odd** coefficients, with the choice **$s = 0$**.

Thus, $y = C_0\, [x + (2 - \lambda)\, x^3 / 3! + (2 - \lambda)(12 - \lambda)\, x^5 / 5! + \cdots]$ ----- (4)

Remember that with $s = 1$, $y = \Sigma\, C_n\, x^{n+1}$, now.

The general solution may now be obtained by combining (iv) with the solution obtained with $s = 0$.

Descending Series :

$$\mathbf{(1 - x^2)\, d^2y/dx^2 - 2x\, dy/dx + \ell(\ell+1)\, y = 0}$$

Here we assume the form of the parameter λ as $\ell(\ell+1)$ and also assume the solution to be a polynomial of order ℓ. So, we take y as : **$y = \Sigma_n\, C_n\, x^{\ell - n}$**

Orthogonality :

Consider two values of the parameter λ, m(m+1) and n(n+1), for which we shall have two different solutions P_m and P_n. So,

$$(1 - x^2)\, d^2P_m /dx^2 - 2x\, dP_m /dx + m(m+1)\, P_m = 0 \quad \text{----- (5)}$$

$$(1 - x^2)\, d^2P_n /dx^2 - 2x\, dP_n /dx + n(n+1)\, P_n = 0 \quad \text{----- (6)}$$

Multiply (5) by P_n and (6) by P_m and subtract. We have :

$$\mathbf{(1 - x^2)(P_n P_m'' - P_m P_n'') - 2x\,(P_n P_m' - P_m P_n')} + \{m(m+1) - n(n+1)\}\, P_m P_n = 0$$

Note that : $d/dx\, (P_n P_m' - P_m P_n') = (P_n P_m'' - P_m P_n'')$, so that we may write :

$$\mathbf{d/dx\, \{(1 - x^2)(P_n P_m' - P_m P_n')\}} + \{m(m+1) - n(n+1)\}\, P_m P_n = 0$$

Now integrate between the limits -1 to $+1$:

$$\mathbf{[}(1 - x^2)(P_n P_m' - P_m P_n')\mathbf{]} + \mathbf{\{}m(m+1) - n(n+1)\mathbf{\}} \int P_m(x) P_n(x)\, dx = 0$$

The first term within square braces vanishes at the limits. So, if $m \neq n$, we must have :

$$\mathbf{\int P_m(x)\, P_n(x)\, dx = 0} \quad \text{----- (7)}$$

You have worked with vectors and you know about 'dot product'. However, the concept of vectors can be generalized much more and also the definition of dot product can be extended. In this generalized sense, even a set of functions may be called vectors and their dot product, or better call them '**inner product**', may be defined as : $\int f_1(x)\, f_2(x)\, dx$, within suitable limits.
In this sense, the above eqn. may be stated as, 'the inner product of P_m and P_n is zero', or in other words, they are '**orthogonal**' (which basically means 'perpendicular'), if $m \neq n$.

Actually this is a property of the so-called **Sturm - Liouville** Problems. The Sturm – Liouville differential equations are of the form :

$$\mathbf{p(x)\, d^2y/dx^2 + p'(x)\, dy/dx = \lambda\, q(x)\, y,}$$

where 'λ' is called an **eigen-value** and the solution $y(x)$ is called the corresponding **eigen-function**. For different values of λ, we have different solutions :

$$p(x)\, d^2y_1/dx^2 + p'(x)\, dy_1/dx = \lambda_1\, q(x)\, y_1 \text{ ----- } (8)$$

$$p(x)\, d^2y_2/dx^2 + p'(x)\, dy_2/dx = \lambda_2\, q(x)\, y_2 \text{ ----- } (9)$$

Multiply (8) by y_2 and (9) by y_1 and subtract. We have :

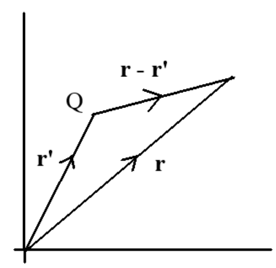
$$p(x)\,(y_2\, y_1'' - y_1\, y_2'') + p'(x)\,(y_2\, y_1' - y_1\, y_2') = (\lambda_1 - \lambda_2)\, q(x)\, y_1\, y_2$$

Again note that : $d/dx\,(y_2\, y_1' - y_1\, y_2') = (y_2\, y_1'' - y_1\, y_2'')$, hence :

$$d/dx\, \{\, p(x)\,(y_2\, y_1' - y_1\, y_2')\, \} = (\lambda_1 - \lambda_2)\, q(x)\, y_1\, y_2$$

If we integrate within such limits where $p(x)$ vanishes then the LHS will disappear and we shall have : $(\lambda_1 - \lambda_2) \int q(x)\, y_1(x)\, y_2(x)\, dx = 0$. If $\lambda_1 \neq \lambda_2$, then $\boldsymbol{\int q(x)\, y_1(x)\, y_2(x)\, dx = 0}$. We say that the eigen-functions corresponding to different eigenvalues are orthogonal. Here of course the inner product is defined as : $\int q(x)\, y_1(x)\, y_2(x)\, dx$.

Generating Function



In Electrostatics, if a charge Q is placed at a point **r'**, the potential due to it at **r** will be : $V(\mathbf{r}) = Q / |\mathbf{r} - \mathbf{r}'| = Q / [\, r^2 + r'^2 - 2rr' \cos\theta]^{½}$

$$= Q/r\, [\, 1 + (r'/r)^2 - 2(r'/r) \cos\theta]^{½},$$

[where θ is the angle between **r** and **r'**.]

$$= (Q/r) \times [\, 1 - \{\, 2(r'/r) \cos\theta - (r'/r)^2\}\,]^{-½}$$

If $r' < r$, we can expand $V(\mathbf{r})$ in a binomial expansion as :

$$V(\mathbf{r}) = Q/r\, [\, 1 + ½\, \{\, 2(r'/r)\cos\theta - (r'/r)^2\} + (-½)(-½ - 1)/2!\, \{\, 2(r'/r)\cos\theta - (r'/r)^2\}^2 + \cdots$$

Arranging in powers of (r'/r) :

$$V(\mathbf{r}) = Q/r\, [\, 1 + (r'/r)\cos\theta + (r'/r)^2\, \{\, 3\cos^2\theta - 1\}/2 + \ldots\,] = Q/r\, \Sigma\, (r'/r)^n\, P_n(\cos\theta)$$

As perhaps you can recognize, the coefficients of $(r'/r)^n$ are the Legendre polynomials.
Motivated by this result, we introduce the **Generating Function** for Legendre polynomials as :

$$\mathbf{G(t, x) = 1/\sqrt{(1 - 2tx + t^2)} = \Sigma\, t^n\, P_n(x), \text{ ----- } (1)}$$

where the variable 't' plays the role of (r'/r) and 'x' that of $\cos\theta$.
Results derived from this definition :

1. **Put $x = 1$** in both sides eqn.(1) : LHS $= 1/\sqrt{(1 - t)^2} = (1 - t)^{-1} = 1 + t + t^2 + t^3 + \ldots$

   RHS $= \Sigma\, t^n\, P_n(1)$

Equating the coefficients of t^n on both sides $\Rightarrow$ **$P_n(1) = 1$**

2. **Replace $t \to -t$, $x \to -x$** in both sides eqn.(1) : LHS = $1/\sqrt{(1 - 2tx + t^2)}$ remains unchanged

$$\Rightarrow \Sigma t^n P_n(x) = \Sigma (-t)^n P_n(-x)$$

Equating the coefficients of t^n again : **$P_n(x) = (-1)^n P_n(-x)$,**

or, equivalently **: $P_n(-x) = (-1)^n P_n(x)$,** since $(-1)^{2n} = 1$.

This implies that the even order Legendre Polynomials are even functions and the odd order ones are odd functions, e.g., $P_1(x) = x$, but $P_2(x) = ½ (3x^2 - 1)$.

3. **Differentiate both sides of G(t, x) w.r.t. 't'** :

LHS = $-½ (1 - 2tx + t^2)^{-3/2} (-2x + 2t) = (1 - 2tx + t^2)^{-3/2} (x - t)$

RHS = $\Sigma nt^{n-1} P_n(x)$

Multiplying both sides by $(1 - 2tx + t^2) \Rightarrow (1 - 2tx + t^2)^{-½} (x - t) = (1 - 2tx + t^2) \Sigma nt^{n-1} P_n(x)$

$$\Rightarrow (x - t) \Sigma t^n P_n(x) = (1 - 2tx + t^2) \Sigma nt^{n-1} P_n(x)$$

Equating the coefficients of t^n : $xP_n(x) - P_{n-1}(x) = (n + 1) P_{n+1}(x) - 2nx P_n(x) + (n - 1) P_{n-1}(x)$

$$\Rightarrow \mathbf{(2n + 1) xP_n(x) - n P_{n-1}(x) = (n + 1) P_{n+1}(x)}$$

The relation provides an expression for P_{n+1} in terms of Pn and P_{n-1}. Hence, by knowing the expressions for P_0 and P_1, we can derive the expression for P_2 and so on.

Rodrigues Formula

There is yet another approach to obtain Legendre Polynomials, viz. the Rodrigues Formula. It defines the polynomials as : **$P_n(x) = 1/(2^n n!) d^n/dx^n (x^2 - 1)^n$.**

$P_0(x) = 1/(2^0 0!) (x^2 - 1)^0 = 1$

$P_1(x) = 1/(2 \times 1!) d/dx (x^2 - 1) = ½ \times 2x = x$

$P_2(x) = 1/(2^2 \times 2!) d^2/dx^2 (x^2 - 1)^2 = (1/8) d/dx \{2 (x^2 - 1) \times 2x\}$

$= ½ d/dx \{x^3 - x\}$

$= ½ \{3x^2 - 1\}$

1. To prove that : $P_n(-x) = (-1)^n P_n(x)$

We start by showing that if f(x) is an even function, its derivative is an odd function.

Example : $x^4 \to 4x^3 \to 12x^2$, etc.

Proof :

Let $f(x) = f(-x)$

Differentiating both sides with respect to x :

$f'(x) = -f'(-x)$.

If we keep on differentiating, we shall find that :

$f^n(x) = (-1)^n f^n(-x)$,

where $f^n(x)$ stands for the n-th derivative of f(x).

Applying this general result for $(x^2 - 1)^n$, which is an even function, we can conclude that :

$P_n(-x) = (-1)^n P_n(x)$.

2. To prove that : $P_n(1) = 1$

We start by showing that the lowest power of $(x^2 - 1)$ in the **m-th** derivative ($m \le n$) of $(x^2 - 1)^n$ is : $(x^2 - 1)^{n-m}$.

$d/dx (x^2 - 1)^n = n (x^2 - 1)^{n-1} \times 2x$

$d^2/dx^2\,(x^2 - 1)^n = n(n-1)\,(x^2-1)^{n-2} \times (2x)^2 + n\,(x^2-1)^{n-1} \times 2$

$d^3/dx^3\,(x^2 - 1)^n = n(n-1)(n-2)\,(x^2-1)^{n-3} \times (2x)^3$ + terms involving higher powers of $(x^2 - 1)$

. . . .

$d^m/dx^m\,(x^2 - 1)^n = n(n-1)(n-2)\,.\,.\,.\,(n-m+1)\,(x^2-1)^{n-m} \times (2x)^m$ + terms involving higher powers of $(x^2 - 1)$

$\Rightarrow\ d^n/dx^n\,(x^2 - 1)^n = n(n-1)(n-2)\,.\,.\,.\,1\,(x^2-1)^{n-n} \times (2x)^n$ + terms involving higher powers of $(x^2 - 1)$

Setting x = 1, this yields : $n!\,2^n \Rightarrow 1/(2^n\,n!)\,d^n/dx^n\,(x^2-1)^n\big|_{x=1} = 1$.

3. To prove that : $\int Pm(x)\,P_n(x)\,dx$ [between limits +1 and – 1] = 0, if $m \neq n$

We drop the consts., because they don't matter in the orthogonality relation :

$$\int Pm(x)\,P_n(x)\,dx = \int D^m (x^2-1)^m\ D^n(x^2-1)^n\,dx$$

Let m < n. Take as the first function.

$$\text{RHS} = [D^m (x^2-1)^m\,D^{n-1}(x^2-1)^n] - \int D^{m+1}(x^2-1)^m\,D^{n-1}(x^2-1)^n\,dx$$

We have shown above that the lowest power of $(x^2 - 1)$ in the **m-th** derivative ($m \leq n$) of $(x^2-1)^n$ is : $(x^2-1)^{n-m}$. Therefore, the lowest power of $(x^2 - 1)$ in the expression for $D^{n-1}(x^2-1)^n$ is 1 $\Rightarrow$ the first term vanishes in the limits $x = \pm 1$. Thus,

$$\text{RHS} = -\int D^{m+1}(x^2-1)^m\,D^{n-1}(x^2-1)^n\,dx$$

$$= -[D^{m+1}(x^2-1)^m\,D^{n-2}(x^2-1)^n] + \int D^{m+2}(x^2-1)^m\,D^{n-2}(x^2-1)^n\,dx$$

The first term vanishes by the same logic

$$\Rightarrow\ \text{RHS} = \int D^{m+2}(x^2-1)^m\,D^{n-2}(x^2-1)^n\,dx$$

. . . .

$$= (-1)^m \int D^{m+m}(x^2-1)^m\,D^{n-m}(x^2-1)^n\,dx,$$

but $(x^2-1)^m$ is an 2m-th order polynomial in x, so $D^{2m}(x^2-1)^m$ is a constant (= 2m!).

$$\Rightarrow\ \text{RHS} \propto \int D^{n-m}(x^2-1)^n\,dx,$$

$$= [D^{n-m-1}(x^2-1)^n],$$

which vanishes in the limits $x = \pm 1$, as we have argued.