Basic Probability

Set: A Collection of distinct and well-defined objects is called a set.

A set is said to be finite set if it has finite number of elements and it is said to be infinite if it has an infinite number of elements.

Example: The set of days in a month is finite and the set of all integer is an infinite set.

A set has no element is called an empty set or null set and is denoted by Ø.

If every element of a set A is also an element of set B, then A is called a subset of set B and is denoted by $A \subset B$.

Random Experiment: An experiment E is called condom experiment, if (I) all possible outcomes of E are known in advance (ii) it is impossible to predict which outcome will occur at a particular performance of E and (iii) E can be repeated under identical conditions infinite number of times i.e. a large number times .

Ex:

(i) The experiment of tossing a coin is example of a random experiment. Here the possible outcomes are head and "tail'. It is impossible to predict which outcome will occur at a particular toss of the coin under given condition.
(ii) Throwing a die,
(iii) Drawing a pack of 52 cards at random.

Performing a random experiment is called a trial Event.

Event Space: The set of all possible outcomes of a given random experiment E is called the event space or sample space of E and is denoted by S.

Ex:

(i) The event space S of the random experiment of tossing a coin is {H,T}, where H and T ,corresponding to the outcomes head and tail respectively . This is an example of a finite event space since it contains a finite number of elements.

The event space S corresponding to the experiment of choosing a number at random from the interval (2,4) is the set (2,4) itself which is an infinite event space.

Event: Let us consider the random experiment of throwing a die its event space is S={1,2,3,4,5,6}.

Here A= {2,4,6} is an event which can be described as 'Even number appears in throwing a die'. The event A happens in a specific trial of the given random experiment if and only if exactly one of the outcomes 2,4 or 6 occur in the trial.

So, an event A of a given random experiment can be defined as a subject of the corresponding event space S.

Impossible event: An event of a given random experiment is called an impossible event if it can never happen in any trial of the random experiment under identical conditions such an event is described by the empty subset Φ of the corresponding event space S.

Ex: In the random experiment 'throwing a die 'the event 'face marked 7 ' is an impossible event.

Certain event: An event of a given random experiment is called 'certain if it happens in every performance of the corresponding random experiment under identical condition.

A certain event contains all the elements of the event space or sample space.

Ex: In 'tossing of a coin ', the event 'H' or 'T' is a certain event.

Simple event: An event A is called a simple event or an elementary event if A contains exactly one element. Tossing a coin is a simple event.

Compound Event: An event A is called a compound event or composite events if A contains more than one element. In other words, events which can be decomposed into to small events are known as compound events or composite events.

A= {1} → simple event

B= {3,6}→ $Compound\ event$

C= {2,4,6}→ $Compound\ event$

Complementary Event: A and $\vec{A}$ are called complementary event if $\vec{A}$ contains elements of the event space which are not in A.

Mutually Exclusive Events: Two events A & B connected to a given random experiment E are said to be mutually exclusive if A &B can never happen simultaneously in any performance of E, i.e. if A ∩ B= AB= Φ i.e. A & b are disjoint

E= throwing of a die = {1,2,3,4,5,6}

A= even face = {2,4,6}

B= odd face = {1,3,5}

C = multiple of 3 = {3,6}

Now A∩C = {6} $\neq \emptyset$ → not mutually exclusive events

B∩C = {3} $\neq \emptyset$ → not mutually exclusive events

A∩B= $\emptyset$ → mutually exclusive events

Exhaustive Events: A collection of events said to be exhaustive if at least one event belonging to the collection is sure to occur in every performance of the underlying random experiment.

E= throwing of a die = {1,2,3,4,5,6}

A= even face = {2,4,6}

B= odd face except 1 = {3,5}

C = {1}

{A, B, C} is exhaustive event as A$\cup B \cup$C= S =certain event.

Equally likely events: If one of the events cannot be expected to occur in preference to another then such events are called equally likely, or equally probable.

Ex: In tossing coin the occurrence of the head or the tail is equally likely.

Classical definition of probability: Let E be a random experiment such that its event space S contains a finite number, say n, of event points, all of which are known to be equally likely. If any event A connected with E contains m(A) number of there event points, then the probability of A, denoted by P(A), will be defined by P(A)= $\frac{m(A)}{n}$

Deductions:

1. 0$\leq P(A) \leq$1

we have P(A)= $\frac{m}{n}$

0$\leq m \leq$n

0$\leq \frac{m}{n} \leq$1

0$\leq P(A) \leq$1

2. P(S) =1
   When A=S, m=n
   P(S)= $\frac{n}{n}$ = 1
3. P(∅)=0

P (∅)= $\frac{0}{n}$ = 0

P ($\vec{A}$) =1- P(A)

P ($\vec{A}$) = $\frac{n-m}{n}$

= 1$-\frac{n}{m}$

= 1$-$P(A)

Problems:

1. A coin is tossed 3 times in succession. Find the probability of (a) 2 heads (b) 2 consecutive heads.
   The total number of points in the event space n = 2^3= 8.
   U_1= (H, H, H) U_2= (H, H, T) U_3= (H, T, H) U_4= (T, H, H) U_5= (T, T, H) U_6= (T, H, T) U_7= (H, T, T) U_8= (T, T, T)

   (a) Let A denotes the event '2 heads'
   A contains 3 event points $U_{2,}\ \ U_{3,}\ U_4$
   i.e., m(A)= 3
   P(A) = $\frac{3}{8}$
   (b) Let B denotes the event '2 consecutive heads'
   B contains 2 event points $U_{2,}\ \ U_4$
   i.e., m(B)= 2
   P(A) = $\frac{2}{8}$ = $\frac{1}{4}$

2. A room has 3 lamps. From a collection of 10 light bulbs of which 5 are defective, a person selects 3 at random and puts them in the sockets. What is the probability that he will have light?

⇨ Total no. of ways of selecting 3 bulbs out of 10 is n =10_{c_3} = $\frac{10!}{3!(10-3)!}$ = 120

Total no. of ways of getting 3 defective lights out of 5 is m = 5_{c_3}= 10

∴ The probability of getting all defective bulbs is = $\frac{10}{120}$ = $\frac{1}{12}$

∴ Probability of light = 1— $\frac{1}{12}$ = $\frac{11}{12}$ (Ans)

3. What is the probability that a leap year, selected at random, will contain 53 Sundays?

⇨ A leap year consist of 366 days that is 52 full weeks (= 354 days) and two extra days These extra two days may be either (SUN, MON), or (MON, TUE), or (TUE, WED), or (WED, THU), or (THU, FRI), or (FRI, SAT), or (SAT, SUN).
∴ A leap year will contain 53 Sundays if one of the extra days is Sunday.
∴ Required Probability = $\frac{2}{7}$ (Ans)

4. A pair of dice is thrown. Find the probability of getting a sum of 7, when it is known that the digit in the first die is greater than that of the second. [WBUT 2009]

⇨ The total no. of outcomes is n = 6^2 = 36

Total no. of event points m = 3

= {(6,1), (5,2), (4,3)}

∴ The required probability = $\frac{m}{n}$ = $\frac{3}{36}$ = $\frac{1}{12}$ (Ans)

5. When two dice are thrown, find the probability that difference od the points on the dice is 2 or 3. [WBUT 2006]

⇨ The total no. of outcomes n = 6^2 = 36

The total no. of event points m = 14

= {(1, 3), (3, 1), (1, 4), (4, 1), (2, 4), (4, 2), (2, 5), (5, 2), (3, 5), (5, 3), (3, 6), (6, 3), (4, 6), (6, 4)}

∴ The required probability = $\frac{m}{n}$ = $\frac{14}{36}$ = $\frac{7}{18}$ (Ans)

6. Two cards are drawn in succession without replacement from a pack of 52 playing cards, Find the probability that
   i) Exactly one of them is spade.
   ii) At least one of them is spade.

⇨ 2 cards may be drawn from 52 cards in ${}^{52}C_2$ ways.

∴ The total no. of outcomes n = ${}^{52}C_2$ = 1326

i) The pack contains 13 spades, so 1 spade may come in ${}^{13}C_1$ ways and the other (not spade) may come ${}^{52-3}C_1$ = ${}^{39}C_1$ ways

∴ The total no. of event points from the event 'exactly one of them is spade' is

${}^{13}C_1 * {}^{39}C_1 = 13 * 39 = 507$

∴ The required probability = $\frac{507}{1326}$ = $\frac{13}{34}$

ii) Let B denotes the event 'at least one drawn card is spade', so B^c denoted the event 'none of two drawn cards is spade'.

∴ The total no. of event points of B^c is

${}^{52-13}C_2 = {}^{39}C_2 = 741$

∴ P (B^c) = $\frac{741}{1326}$ = $\frac{19}{34}$

∴ P (B) = $1 - \frac{19}{34}$ = $\frac{15}{34}$ (Ans)

7. Two cards are drawn in succession from a pack of 52 cards. Find the chance (probability) that the first is a king and the second a queen, if the first card is (i)replaced, (ii)not replaced.

⇨

i) In the case of first card is replaced the total no. of outcomes = 52 $\times$52

Now, the pack consists of 4 kings and 4 queens.

So, the 1st king card comes in $^4C_1 = 4$ ways and the 2nd queen card may come in $^4C_1 = 4$ ways.

∴ The required probability = $\frac{4\times4}{52\times52} = \frac{1}{169}$

ii) In case of first card is not replaced, the total no. of outcomes = 52×51

∴ The required probability = $\frac{4\times4}{52\times51} = \frac{4}{13\times51} = \frac{4}{663}$

8. A bag contains 6 red, 4 white and 8 blue balls. If 3 balls are drawn at random, find the probability that one is red, one is white and one is blue.

⇨ Total no. of outcomes = $^{18}C_3 = 816$

Total no. of event points = $^6C_1 \times {}^4C_1 \times {}^8C_1 = 6 \times 4 \times 8 = 192$

∴ The required probability = $\frac{192}{816} = \frac{4}{17}$ (Ans)

Axiomatic Definition of Probability:

Let E be a given random experiment and S be the corresponding event space.
Let A be an event associated with the random experiment E, then probability of A denoted by P (A), is a real number which satisfies the following axioms:

A1 => $P(A) \geq 0$ for every $A \in S$

A2 => $P(S) = 1$,

A3 => if $A_1, A_2, \ldots, A_n, \ldots$ be a finite or infinite number of pairwise mutually exclusive events, i.e., $A_i, A_j \in S$, then $P(A_1 + A_2 + \ldots + A_n + \ldots)$

$$= P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$$

The entire mathematical structure of the theory of probability can be logically built up on the basis of these axioms.

Deductions from Axiomatic Definition:

Th1: Probability of an impossible event is zero, i.e., $P(\Phi) = 0$

Proof=> Now, $S + \Phi = S$

Both S & Φ are disjoint, i.e., $S\Phi = \Phi$

$P(S + \Phi) = P(S)$

⇨ $P(S) + P(\Phi) = P(S)$

⇨ $P(\Phi) = 0$

Th2: Probability of the complementary event $\overline{A}$ of A is $P(\overline{A}) = 1 - P(S)$. Also, $0 \leq P(A) \leq 1$

Proof=> A & $\overline{A}$ are disjoint events.

$\therefore A + \overline{A} = S$

$P(A + \overline{A}) = P(S)$ [by A3]

⇨ $P(A) + P(\overline{A}) = P(S) = 1$ [by A1]

⇨ $P(\overline{A}) = 1 - P(A)$

⇨ $0 \leq P(A) \leq 1$ [by A1]

Th3=> For any two events A & B,

$P(\overline{A}B) = P(B) - P(AB)$

Proof=> Now, AB & $\overline{A}B$ are disjoint.

$AB + \overline{A}B = B$

⇨ $P(AB + \overline{A}B) = P(B)$

⇨ $P(AB) + P(\overline{A}B) = P(B)$ [By A3]

⇨ $P(\overline{A}B) = P(B) - P(AB)$

Th4(Addition Theorem) => For any two events A and B (may not be mutually exclusive)

$P(A+B) = A + \overline{A}B$

- ⇨ $P(A+B) = P(A+\bar{A}B)$

$= P(A) + P(\bar{A}B)$ [By A3]

$= P(A) + P(B) - P(AB)$ [By Th3]

Th5 (Extension of Th4) => For any three events A, B, C,

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$$

L.H.S $= P(A+B+C)$

$= P\{(A+B) + C\}$

$= P(A+B) + P(C) - P\{(A+B)C\}$

$= P(A) + P(B) + P(C) - P(AC+BC) - P(AB)$

$= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

Deduction of Classical Definition:

Let the event space (of a random experiment E) contains n district event points (i.e., simple events) $u_1, u_2, \ldots, u_n$

$\therefore u_1 + u_2 + \ldots + u_n = S$

- ⇨ $P(u_1 + u_2 + \ldots + u_n) = P(S)$
- ⇨ $P(u_1) + P(u_2) + \ldots + P(u_n) = 1$

Let the simple events have equal probability,

i.e., $P(u_1) = P(u_2) = \ldots = P(u_n) = \frac{1}{n}$

Now, suppose A be an event connected to the given random experiment E.

If A contains m ($\leq$ n) event points of S, then without loss of generality, we can write,

$A = u_1 + u_2 + \ldots + u_m$

$\therefore P(A) = P(u_1 + u_2 + \ldots + u_m)$

$= P(u_1) + P(u_2) + \ldots + P(u_m)$

$= \frac{1}{n} + \frac{1}{n} + \frac{1}{n}$

$= \frac{m}{n} = \frac{m(A)}{n}$,

which is the classical definition of probability deduced from axiomatic definition of probability.

Th6: For any two events A and B,

$P(AB) \le P(A) \le P(A+B) \le P(A) + P(B)$

Proof=> Now, AB, $A\bar{B}$ are disjoint and AB+ $A\bar{B}$ = A

$\therefore P(AB + A\bar{B}) = P(A)$

⇨ $P(AB) + P(A\bar{B}) = P(A)$

Since $0 \le P(A\bar{B}) \le 1$ so P(AB) can never exceed P(A)

i.e., $P(AB) \le P(A)$ ------- (i)

Again, $P(A+B) = P(A) + P(B) - P(AB)$

$= P(A) + P(AB)$ --------(ii)

$0 \le P(A\bar{B}) \le 1$, so $P(A) \le P(A+B)$

Now, $P(A+B) = P(A) + P(B) - P(AB) \le P(A) + P(B)$ $[0 \le P(AB) \le 1]$ ----- (iii)

Note: Similarly, $P(AB) \le P(B) \le P(A) + P(B)$

Th7(Boole's Inequality) =>

For any n events $A_1, A_2, \ldots, A_n$ connected to a random experiment E,

$P(A_1 + A_2 + \ldots + A_n) \le P(A_1) + P(A_2) + \ldots + P(A_n)$

Proof=> We have for any two events A_1, A_2,

$P(A_1 + A_2) = P(A_1) + P(A_2) - P(A_1A_2) \le P(A_1) + P(A_2)$ $[P(A_1A_2) \ge 0]$ ------(i)

$\therefore$ the given inequality is true for n = 1,2

Let us assume that the given inequality is true for any +ve integer m≥2

i.e., $P(A_1 + A_2 + \ldots + A_m) \le P(A_1) + P(A_2) + \ldots + P(A_m)$

$\therefore P(A_1 + A_2 + \ldots + A_m + A_{m+1}) = P[(A_1 + A_2 + \ldots + A_m) + A_{m+1}]$

$\le P(A_1 + A_2 + \ldots + A_m) + P(A_{m+1})$

$\le P(A_1) + P(A_1) + \ldots + P(A_m) + P(A_{m+1})$

Thus, the given inequality is true for m+1 whenever it is true for m. But we have already seen that this inequality is true for 2, hence it is true for 2+1=3, 3+1=4, 4+1=5, etc. Hence the given inequality is true for any +ve integer n.

Th8(Bonferroni's Inequalities) => For any n events $A_1, A_2, \ldots, A_n$ connected to a random experiment E

i) $P(A_1A_2 \ldots A_n) \ge 1 - \sum_{l=1}^{n} P(\bar{A}_i)$
ii) $P(A_1A_2 \ldots A_n) \ge 1 - \sum_{i=1}^{n} P(\bar{A}_i) - (n-1)$ [WBUT 2003]

Proof => i) We have by Boole's inequality,

$$P(\bar{A}_1 + \bar{A}_2 + \ldots + \bar{A}_n) \le P(\bar{A}_1) + P(\bar{A}_1) + \ldots + P(\bar{A}_n)$$

- ⇨ $P(\overline{A_1 A_{12} \ldots A_n}) \le P(\bar{A}_1) + P(\bar{A}_1) + \ldots + P(\bar{A}_n)$ [By De Morgan's Law]
- ⇨ $1 - P(A_1A_2 \ldots A_n) \le \sum_{i=1}^{n} P(\bar{A}_i)$
- ⇨ $1 - \sum_{i=1}^{n} P(\bar{A}_i) \le P(A_1A_2 \ldots A_n)$
- ⇨ $P(A_1A_2 \ldots A_n) \ge 1 - \sum_{i=1}^{n} P(\bar{A}_i)$

ii) We know that,

$$P(\bar{A}_i = 1 - P(A_i)$$

$$\therefore 1 - \sum_{i=1}^{n} P(\overline{A_i}) = 1 - \sum_{i=1}^{n} [1 - P(A_i)] = \sum_{i=1}^{n} P(A_i) - (n-1)$$

Using (i), $P(A_1A_2 \ldots A_n) \ge \sum_{i=1}^{n} P(\bar{A}_i) - (n-1)$

Conditional Probability:

Let A and B are any two events connected to a given random experiment E. The conditional probability of the event A on the hypothesis that the event B has occurred is denoted and defined by

$$P(A/B) = \frac{P(AB)}{P(B)}, \text{ provided } P(B) \neq 0$$

Similarly, we can define,

$$P(B/A) = \frac{P(AB)}{P(A)}, \text{ provided } P(A) \neq 0$$

Example 1: Let E be the random experiment of throwing of a die with event space S. Let A denoted the event 'even face' and B denoted the event 'multiple of 3'.

⇨ S = {1,2,3,4,5,6}, A = {2,4,6}, B = {3,6},

$\therefore AB = A \cap B = \{6\}$

$\therefore P(A) = \frac{3}{6} = \frac{1}{2}$, $\quad P(B) = \frac{2}{6} = \frac{1}{3}$

$\therefore P(A/B) = \frac{P(AB)}{P(B)} = = \frac{1/6}{1/3} = \frac{3}{6} = \frac{1}{2}$

$$\therefore P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{2}{6} = \frac{1}{3}$$

Example 2: Let A and B are two events such that $P(A \cup B) = \frac{7}{8}$, $P(A \cap B) = \frac{1}{4}$, and $P(A^c) = {}^5/_8$. Find the conditional probability of A on the hypothesis that B does not occur. [WBUT 2003]

⇨ $P\left({}^{A}/_{B^C}\right) = \frac{P(A \cap B^C)}{P(B^C)} = P(A) - \frac{P(A \cap B)}{1 - P(B)}$

$P(A) = 1 - P(A^C) = 1 - \frac{5}{8} = \frac{3}{8}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{7}{8} = \frac{3}{8} + \frac{1}{4}$$

$$P(B) = \frac{7}{8} - \frac{3}{8} + \frac{1}{4} = \frac{7 - 3 + 2}{8} = \frac{6}{8} = \frac{3}{4}$$

∴ Required probability = $P\left(\frac{A}{B^c}\right) = \frac{3}{8} - \frac{\frac{1}{4}}{1 - \frac{3}{4}}$

$= \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$ (Ans)

Baye's Theorem:

If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive events one of which certainly occurs (i.e., mutually exhaustive) and let B be any event connected to the same random experiment provided that the conditional probabilities $P(B/A_r)$, r=1, 2, …, n are defined then,

$$P(A_i B) = \frac{P(A_i)P(B/A_i)}{\sum_{r=1}^{n} P(A_r)P(B/A_r)}$$

Proof=> Here $A_iA_j = \Phi$, $i \neq j$, i, j = 1, 2, …, n

$A_1 + A_2 + \ldots + A_n$ = certain event = S, since one of $A_1, A_2, \ldots, A_n$ certainly occurs

∴ $B = SB = (A_1 + A_2 + \ldots + A_n) B$

$= A_1B + A_2 B + \ldots + A_n B$

Also, $(A_iB)(A_jB) = A_iA_j) B = \Phi B = \Phi$ ($I \neq j$, i,j = 1,2,…,n)

∴ $P(B) = P(A_1B) + P(A_2 B) + \ldots + P(A_n B)$

$= \sum_{i=1}^{n} P(A_i B)$

$= \sum_{i=1}^{n} P(A_i) P\left(\frac{B}{A_i}\right)$ $\left[\therefore P\left(\frac{B}{A_i}\right) = \frac{P(A_iB)}{P(A_i)}\right]$

$$\therefore P(B) = \sum_{i=1}^{n} P(A_i) P\left(\frac{B}{A_i}\right) \quad \text{---- (i)}$$

This theorem is known as the theorem on total probability.

By Definition; $\frac{P(A_i)P\left(\frac{B}{A_i}\right)}{\sum_{r=1}^{n} P(A_r)P\left(\frac{B}{A_r}\right)}$ [pd]

1. A bag contains 4 white and 2 black balls and a second bag contains 3 of each color. A bag selected a random and the ball is then taken out at random from the back chosen what is the probability that the ball selected is a white?

⇨ Let A_i (i=1,2) be the event of select in i^{th} bag and B be the event of selecting a white ball.

$P(A_1) = P(A_2) = \frac{1}{2}$

$P\left(\frac{B}{A_1}\right) = \frac{4}{6} = \frac{2}{3}, P\left(\frac{B}{A_2}\right) = \frac{3}{6} = \frac{1}{2}$

$\therefore P(B) = P(A_1) P(B/A_1) + P(A_2) P(B/A_2)$

$= \frac{1}{2}.\frac{2}{3} + \frac{1}{2}.\frac{1}{2} = \frac{7}{12}$

2. 3 identical boxes I, II and III contain respectively 4 white and 3 red balls, 3 white and 7 red balls, 2 white and 3 red balls. A box is chosen and random and a ball is drawn out of it if the ball is found to be white what is the probability that box II is selected. [WBUT 2007]

⇨ Let A_1, A_2 and A_3 denote the events at the ball is drawn from boxes I, II and III respectively, clearly the events A_1, A_2 and A_3 and mutually exclusive.

$\therefore P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

Also let A denote the event that the drawn ball is white.

$\therefore P(A/A_1) = \frac{4}{7}$, $\therefore P(A/A_2) = \frac{3}{10}$, $P(A/A_3) = \frac{2}{5}$

Using Baye's theorem, the required probability is

$$= P(A_2/A) = \frac{P(A_2)P\left(\frac{A}{A_2}\right)}{P(A_1)P\left(\frac{A}{A_1}\right)+P(A_2)P\left(\frac{A}{A_2}\right)+P(A_3)P(A/A_3)} = \frac{\frac{1}{3}.\frac{3}{10}}{\frac{1}{3}.\frac{4}{7}+\frac{1}{3}.\frac{3}{10}+\frac{1}{3}.\frac{2}{5}} = \frac{21}{89}$$

3. The probabilities of x, y and z becoming principal of a college or respectively 0.3, 0.5 and 0.2 the probability that 'Student Aid-Fund' will be introduced in the college if X y and z become principal are 0.4, 0.6 and 0.1 respectively given that 'Student Aid-Fund' has been introduced find the probability that why has been appointed as the principal.
[WBUT 2004]

⇨ Let A, B and C respectively denote the events that x, y and z are becoming the principal and F denotes the event that 'Student Aid-Fund' is introduced.

$\therefore P(A) = 0.3, P(B) = 0.5, P(C) = 0.2$

$P(f/A) = 0.4, P(F/B) = 0.6, P(F/C) = 0.1$

$$\therefore P(B/F) = \frac{P(B)P\left(\frac{F}{B}\right)}{P(A)P\left(\frac{F}{A}\right)+P(B)P\left(\frac{F}{B}\right)+P(C)P\left(\frac{F}{C}\right)} = \frac{0.5*0.6}{0.3*0.4+0.5*0.6+0.2*0.1} = \frac{15}{22}$$

4. In a bold factory machines A, B, C manufacture respectively 25% 35% and 40% of the total of their output 5%, 4%, 2 % are defective bolts. A Bolt is drawn and random from their product and is found to be defective what are the probabilities that it was manufactured by machines A, B and C?

⇨ Let E_1, E_2, E_3 denote the events that are bold drawn at random is manufactured by the machines A, B, C respectively and D denotes the event that the bolt is defective.

Given, $P(E_1) = 0.25$, $P(E_2) = 0.35$, $P(E_3) = 0.40$

$P(D/E_1)$ = probability of drawing a defective Bolt manufactured by A = 0.05

$P(D/E_2) = 0.04$, $P(D/E_3) = 0.02$

$$P(E_1/D) = \frac{P(E_1)P\left(\frac{D}{E_1}\right)}{\sum_{i=1}^{3} P(E_i)P\left(\frac{D}{E_i}\right)} = \frac{0.25*0.05}{0.25*0.05+0.35*0.04+0.4*0.02} = \frac{125}{345} = \frac{25}{69}$$

$$P(E_2/D) = \frac{P(E_2)P\left(\frac{D}{E_2}\right)}{\sum_{i=1}^{3} P(E_i)P\left(\frac{D}{E_i}\right)} = \frac{140}{345} = \frac{28}{69}$$

$$P(E_3/D) = 1 - \left\{P\left(\frac{E_1}{D}\right) + P\left(\frac{E_2}{D}\right)\right\} = 1 - \left(\frac{25}{69} + \frac{28}{69}\right) = \frac{16}{69}$$

5. In answering a question on a multiple-choice test, a student either knows the answer or he guesses, let p be the probability that he knows the answer and 1 – p be the probability that he guesses. Assume that a student who gaseous the answer will be correct with probability $\frac{1}{5}$ what is the condition of probability that a student new the answer to a question given that he answers it correctly?

⇨ Let E_1 be the event that examinee knows the answer, E_2 be the event that examine answers correctly.

$P(E_1) = p$, $P(E_2) = 1-p$, $(A/E_1) = 1$, $P(A/E_2) = 1/5$

Required probability = $P(E_1/A) = \frac{P(E_1)P\left({}^{A}/_{E_1}\right)}{P(E_1)P\left({}^{A}/_{E_1}\right)+P(E_2)P\left({}^{A}/_{E_2}\right)} = \frac{p.1}{p.1+(1-p).\frac{1}{5}} = \frac{5p}{1+4p}$

Independent Events:

Let A and B are two events connected to a given random experiment E.

If P(A/B) = P(A), where P(B) ≠ 0, then we can say that the probability of A does not depend on the happening of B, i.e., the events A and B are independent. Similarly, if P(B/A) = P(B), where P(A) ≠0 then we can say that the probability of B does not depend on the occurrence of A.

∴ If A, B are two independent events,

Then P(A/B) = $\frac{P(AB)}{P(B)}$

⇔ P(AB) = P(A)P(B)

$$P\left({}^{B}/_{A}\right) = \frac{P(AB)}{P(A)}$$

Multiplication theorem:

To events A and B are independent event only if P(Ab) = P(A)P(B)

Defn 1: two events a and b are said to be statistically independent or simply independent if and only if P(AB) = P(A)P(B)

Note: If P(AB) ≠ P(A)P(B), then A and B are said to be dependent.

Defn 2: Three events A, B, C are said to be pairwise independent if

P(AB) = P(A)P(B)

P(BC) = P(B)P(C)

P(CA) = P(C)P(A)

Defn 3: Three events A, B, C are said to be mutually independent if

P(AB) = P(A)P(B)

P(BC) = P(B)P(C)

P(CA) = P(C)P(A)

P(ABC) = P(A)P(B)P(C)

1. If A and B are two events such that P $(A^C \cup B^C) = \frac{5}{6}$, P(A) = $\frac{1}{2}$ and $P(B^C) = \frac{2}{3}$. Show that A and B are independent. [WBUT 2004]

⇨ P $(A^C \cup B^C) = 1 - \frac{5}{6} = \frac{1}{6}$

P(A ∩ B) = $\frac{1}{6}$ ---(i) [By De Morgan's Law]

Also, P(B) = 1-P$(B^C) = 1 - \frac{2}{3} = \frac{1}{3}$

Now, P(A). P(B) = $\frac{1}{2}.\frac{1}{3} = \frac{1}{6}$ ---- (ii)

From (i) & (ii), we get,

P (A ∩ B) = P(AB) = P(A). P(B)

∴ A & B are independent [pd]

2. Suppose the following two boxes are given box A contains 3 red and 2 white marbles, box B contains 2 red and 5 white marbles. A box is selected at random; a marble is drawn and put into the other box then a marble is drawn from the second box, find the probability p that both marbles drawn are of the same color.

⇨ $p = \frac{1}{2}.\frac{3}{5}.\frac{3}{8} + \frac{1}{2}.\frac{2}{5}.\frac{3}{4} + \frac{1}{2}.\frac{2}{7}.\frac{2}{3} + \frac{1}{2}.\frac{5}{7}.\frac{1}{2} = \frac{901}{1680} \approx 0.536$

3. Let X, Y, Z be 3 coins in a box suppose X is a fair coin, Y is two-headed and Z is weighted so that the probability of heads is 1/3. A coin is selected at random and is tossed.
   a. Find the probability that heads appear, that is, Find P(H)
   b. If heads appear, find the probability that is the fair coin X, that is, find P(X/H)
   c. If tails appear, find the probability it is the coin Z, that is, find P(Z/T)

⇨ a) P(H) = $\frac{1}{3}.\frac{1}{2} + \frac{1}{3}.1 + \frac{1}{3}.\frac{1}{3} = \frac{11}{18}$ (Ans)

⇨ b) P(X/H) = $\frac{P(XH)}{P(H)} = \frac{\frac{1}{6}}{\frac{11}{18}}$

P(H/X) = $\frac{1}{2} = \frac{P(XH)}{P(X)} = \frac{P(XH)}{\frac{1}{3}} = \frac{18}{6*11} = \frac{3}{11}$ (Ans)

∴ P(XH) = $\frac{1}{2}.\frac{1}{3} = \frac{1}{6}$

⇨ c) P(Z/T) = P(ZT)/P(T) = $\frac{\frac{2}{9}}{\frac{7}{18}} = \frac{2}{9} \times \frac{18}{7} = \frac{4}{7}$ (Ans)

P(T/Z) = $1 - \frac{1}{3} = \frac{2}{3} = \frac{P(ZT)}{P(Z)} = \frac{P(ZT)}{\frac{1}{3}}$

∴ P(ZT) = $\frac{2}{3}.\frac{1}{3} = \frac{2}{9}$

P(T) = $\frac{1}{3}.\frac{1}{2} + \frac{1}{3}.0 + \frac{1}{3}.(1 - \frac{1}{3}) = \frac{1}{6} + \frac{1}{3}.\frac{2}{3} = \frac{1}{6} + \frac{2}{9} = \frac{7}{18}$