Binomial Distribution

Let (i) there are n independent trials in a random experiment, (ii) Each trial has exactly two mutually exclusive outcomes, namely success and failure, (iii) the probability of success is p and the probability of failure is q in a single trial so that $p + q = 1$ and (iv) X denotes a random variable representing the number of successes in this n trials.

There r successes can be obtained in n trials in n_{c_r} ways and

P (X=r) = $n_{c_r} p^r q^{n-r}$ ……………………………………… (1)

where 0≤ $p \leq 1, p + q = 1$, r= 0,1,2,,,,,,,,, n

The probability distribution (1) is called the binomial probability and X is called the binomial variate denoted as X$\sim B(n, p)$, where n, p are called the parameters of the binomial distribution.

So, a discrete random variable X is said to have binomial distribution with parameters p (0≤ $p \leq 1$) and n (a positive integer) if its distribution is given by

X= i: 0 1 2 3……………… n

P(x=i) = $f_i : f_0\ f_1\ f_2\ f_3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots . f_n$

Where the p.m.f = f_i = P(x=i) = $n_{c_i} p^i (1 - p)^{n-i}$, i= 0,1,2,,,,,,,,, n.

Observe that , $f_i \geq 0$ for all i

And $\sum_{i=0}^{n} f_i$ = $\sum_{i=o}^{n} n_{c_i} p^i (1-p)^{n-i}$ = $\{(1 - p) + p\}^n$ [by binomial expansion]

= 1

So this is a valid probability distribution.

1. If 5 % of bolts produced by a machine are defective, find the probability that out of 10 bolts (drawn at random) (i) none (ii) one (iii) at most 2 bolts will be defective.

   Solution: Given probability of defective bolts = p= $\frac{5}{100}$= 0.05
   So, the probability of non-defective bolts q= 1-0.05= 0.95,
   Total no. of bolts 10.

(i) Probability that none is defective P (0) = $10_{c_0} p^0 q^{10}$= $(0.95)^{10}$=0.599

(ii) Probability of 1 defective P (1) = $10_{c_1} p^1 q^9$= $10 \times 0.05 \times (0.95)^9$= 0.315

(iii) Probability of at most 2 defective P (0) +P (1) + P (2) = $0.599 + 0.315 + 0.0746$ = 0.9886

2. The overall percentage of failures in a certain examination is 40. What is the probability that out of a group of 6 candidates at least 4passed the examination?

Solution: P (X≥ 4) = P(X=4) + P(X=5) + P(X=6)

= $6_{c_4} (0.6)^4 (0.4)^{6-4}$ + $6_{c_5} (0.6)^5 (0.4)^{6-5}$+ $6_{c_6} (0.6)^6 (0.4)^{6-6}$= 0.54432

Recurrence or recursion formula for the binomial distribution

In a binomial distribution P (X=r) =P(r) = $n_{c_r} p^r q^{n-r} = \frac{n!}{r!\times(n-r)!} p^r q^{n-r}$

P (X=r+1) =P(r+1) = $n_{c_{r+1}} p^r q^{n-r} = \frac{n!}{(r+1)!\times(n-r-1)!} p^{r+1} q^{n-r-1}$

Now $\frac{\mathrm{P(r+1)}}{\mathrm{P(r)}} = \frac{n!}{(r+1)!\times(n-r-1)!} \times \frac{r!\times(n-r)!}{n!} \times \frac{p}{q}$

$= \frac{(n-r)}{(r+1)} \cdot \frac{p}{q}$

Mean and variance of the Binomial Distribution

Theorem: If the random variable X has binomial distribution with parameters n and p, then (i) mean = E(X)= np, and (ii) Var(x) = npq, where q=1-p

Given X$\sim B(n,p)$

P (X=r) =P(r) = $n_{c_r} p^r q^{n-r}$ where 0$\leq p \leq 1, p + q = 1$, r= 0,1,2,,,,,,,,, n

Mean = E(X)=$\sum_{r=0}^{n} rP(r)$ =$\sum_{r=1}^{n} r\, n_{c_r} p^r q^{n-r}$ = np $\sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\times(n-r)!} p^{r-1} q^{n-r}$

= np $\sum_{r=1}^{n} n-1_{c_{r-1}} p^{r-1} q^{n-r}$= np $\sum_{r=0}^{n-1} n-1_{c_r} p^r q^{n-1-r}$

=np$(p+q)^{n-1}$= np

(ii) E{x(x−1)} = $\sum_{r=0}^{n} r(r-1)P(r) = \sum_{r=2}^{n} r(r-1)\, n_{c_r} p^r q^{n-r} = \sum_{r=2}^{n} r(r-1) \frac{n!}{r!\times(n-r)!} p^r q^{n-r}$

=n(n−1)$p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\times(n-r)!} p^{r-2} q^{n-r}$

=n(n−1)$p^2 \sum_{r=0}^{n-2} \frac{(n-2)!}{r!\times(n-2-r)!} p^r q^{n-2-r}$

=n(n−1)$p^2 \sum_{r=0}^{n-2} n-2_{c_r} p^r q^{n-2-r}$

=n(n−1)$p^2 (p+q)^{n-2}$ =n(n−1)p^2

Var(X) = E{x(x−1)}−m(m−1) = n(n−1)p^2 −np(np−1) = np−np^2=np (1−p)=npq.

Standard deviation (S. d) =$\sqrt{npq}$

Prob.1 The mean and standard deviation of a Binomial distribution are respectively 4 and $\sqrt{\frac{8}{3}}$. Find (i) n and p, (ii) P(X=0)

Solution:

Since np=4

$\text{npq} = \frac{8}{3} \Rightarrow 4\text{q}= \frac{8}{3} \Rightarrow q = \frac{2}{3}$

p=$1-q = \frac{1}{3}$

Again n= $\frac{4}{p}$ = 12

$\text{P(X=0)} = n_{c_0} p^0 q^{n-0} = q^n = (\frac{2}{3})^{12}$

Prob2: A and B play a game in which their chances of winning are in the ratio 3:2. Find A's chances of winning at least three games out of the five games played.

Solu:

$\Rightarrow \frac{p}{q} = \frac{3}{2} \Rightarrow q = \frac{2p}{3}$

Again p+q= 1 $\Rightarrow p + \frac{2p}{3} = 1 \Rightarrow$ p=$\frac{3}{5}$ and q=$\frac{2}{5}$

Required probability

= P(X=3) + P(X=4) + P(X=5)

= $5_{c_3}(\frac{3}{5})^3(\frac{2}{5})^2 + 5_{c_4}(\frac{3}{5})^4(\frac{2}{5})^1 + 5_{c_5}(\frac{3}{5})^5(\frac{2}{5})^0 = \frac{1}{55}[\frac{1}{2} \times 5 \times (5-1) \times 108 + 5 \times 162 + 243] = \frac{2133}{3125} =$ 0.68256

Prob3: Six dice are thrown 729 times. How many times do you except at least three dice to show a 5 or 6.

Solu:

p=$\frac{2}{6} = \frac{1}{3}$ and q=$\frac{2}{3}$

P (X=r) = $6_{c_r}(\frac{1}{3})^r(\frac{2}{3})^{6-r}$ .... (1)

where $0 \leq p \leq 1, p + q = 1$, r= 0,1,2,3,4,5,6

P (X$\geq$ 3)= 1$-$[P(X=0) + P(X=1) + P(X=2)]

= $1-[6_{c_0}(\frac{1}{3})^6(\frac{2}{3})^0 + 6_{c_1}(\frac{1}{3})^1(\frac{2}{3})^5 + 6_{c_2}(\frac{1}{3})^2(\frac{2}{3})^4] = 1 - \frac{496}{729} = \frac{233}{729}$

Expected no =$729 \times \frac{233}{729} = 233$