Classification

The highest order of derivative appearing in a differential equation is referred to as its **order**.

Ex : $\mathbf{d^2y/dx^2 + (dy/dx)^3 + y} = 0$ – is a second order differential equation.

The power of the highest order derivative term is known as its **degree**.

The degree of the above differential equation is 1.

Ex : $\mathbf{(dy/dx)^3 + y = 0}$

The order of the above differential equation is 1, but its degree is 3.

If the powers of the dependent variable and all its derivatives are '1', the differential equation is called **linear**, otherwise it is non-linear.

Ex : $\mathbf{d^2y/dx^2 + 2x\, dy/dx + ny = x^2}$ – is a linear differential equation, but the equation :

$\mathbf{d^2y/dx^2 + 2x\, dy/dx + ny = y^2}$ – is non-linear.

If the differential equation has a term, independent of the dependent variable and its derivatives, the differential equation is called **inhomogeneous**. This term is usually placed on the right hand side and called the source term. If no such term is present, the equation is homogeneous.

Ex : the diff. eqn. of a forced, damped harmonic oscillator :

$\mathbf{m\, d^2x/dt^2 + 2b\, dx/dt + kx = A \sin \omega t}$ – is inhomogeneous and the source term is the RHS.

the diff. eqn. of a free, damped harmonic oscillator :

$m\, d^2x/dt^2 + 2b\, dx/dt + kx = 0$ – is homogeneous.

The diff. eqn. $\mathbf{m\, d^2x/dt^2 + 2b\, dx/dt + kx = \alpha x^2}$ – is however not inhomogeneous; rather, it is non-linear and usually written as : $m\, d^2x/dt^2 + 2b\, dx/dt + kx - \alpha x^2 = 0$.

If the coefficients of the dependent variable and all its derivatives are constants, the differential equation is said to have **constant coefficients**.

Ex : $\mathbf{d^2x/dt^2 + 4\, dx/dt + 5x = 0}$ – is a differential equation with constant coefficients.

$\mathbf{x^2\, d^2y/dx^2 + 2x\, dy/dx + 5y = 0}$ – is a differential equation with non-constant coefficients.

$\mathbf{d^2y/dx^2 + 2y\, dy/dx + 5y = 0}$ – is however a non-linear differential equation.

First order Liner Equations :

The most general form of such equations is : $a(x)\, dy/dx + b(x)\, y + c(x) = 0$, which may be recast in the form :

$$\mathbf{dy/dx + P(x)\, y = Q(x),} \quad \text{provided of course, } a(x) \neq 0.$$

Let us first consider the simple case where P(x) is a constant = p.

Note that : $d/dx\, \{e^{px}\, y\} = : e^{px}\, dy/dx + p\, e^{px}\, y = e^{px}\, \{dy/dx + py\}$

Thus, if we multiply $\{dy/dx + py\}$ by e^{px}, the result becomes a total derivative of $\{e^{px}\, y\}$.

The factor e^{px} is called an **Integrating Factor** (I.F.).

In a more general case, where P(x) is not a constant, $\mathbf{I.F. = \exp\{\int P(x)\, dx\}}$. Let's take an example :

Find the I.F. for : $dy/dx + x\, y$

Clearly, $P(x) = x$, here. So. $\int P(x)\, dx = x^2/2 \Rightarrow$ I.F. $= \exp(x^2/2)$. Now check :

$\exp(x^2/2)\, dy/dx + \exp(x^2/2)\, x\, y = d/dx\, \{\exp(x^2/2)\, y\}$

How does it help ? Take another example :

Solve : $dy/dx + y/x = x^n$

I.F. = $\exp\{\int 1/x\, dx\} = \exp\{\ln x\} = x$. Multiplying both sides by x : $x\, dy/dx + y = x^{n+1}$

LHS is expected to become a total derivative, which indeed it is ! $x\, dy/dx + y = d/dx\, (xy)$

Thus, $d/dx\, (xy) = x^{n+1} \Rightarrow xy = x^{n+2}/(n+2) + C \Rightarrow y = x^{n+1}/(n+2) + C/x$

Another example : Solve $dy/dx + xy = x$

I.F. = $\exp\{\int x\, dx\} = \exp\{x^2/2\}$. Multiplying both sides by I.F. :

$$\exp\{x^2/2\}\, dy/dx + x \exp\{x^2/2\}\, y = x \exp\{x^2/2\}$$

$$\Rightarrow d/dx\, \{\exp(x^2/2)\, y\} = x \exp\{x^2/2\}$$

$$\Rightarrow \{\exp(x^2/2)\, y\} = \int x \exp\{x^2/2\}\, dx.$$

RHS can be integrated by substituting $x^2/2 = z \Rightarrow x\, dx = dz$,

so that RHS becomes : $\int e^z\, dz = e^z = \exp\{x^2/2\}$

Thus, $\{\exp(x^2/2)\, y\} = \exp\{x^2/2\} + C \Rightarrow y = 1 + C \exp\{-x^2/2\}$

- Do not forget to include the arbitrary constant 'C'. It's a small crime in case of ordinary integration, but a serious crime in case of a differential equation. Besides, you may miss a major part of the solution (not just a constant term), as case of the example above.
- One might ask why we skipped 'C' while finding the I.F. then ? The answer is : it doesn't matter ! $\exp\left\{\int P(x)\, dx + C\right\} = e^C \exp\left\{\int P(x)\, dx\right\} = A \exp\left\{\int P(x)\, dx\right\}$ and the constant 'A' drops from both sides.
- One might worry that after multiplying by the I.F., the RHS may become difficult to integrate. This indeed may happen, but that is an *integration problem*. We have essentially solved the differential equation ! Technically, one says, 'we have reduced it to quadrature'.

Second Order, Homogeneous, Linear Differential Equation with Constant Coefficients

The differential eqn. is of the form :

$$\mathbf{a\, d^2y/dx^2 + b\, dy/dx + c\, y = 0} \quad \text{---- (1)}$$

Try : $y = e^{mx} \Rightarrow dy/dx = m\, e^{mx} \Rightarrow d^2y/dx^2 = m^2\, e^{mx}$

Substituting in the differential eqn. we get : $(a\, m^2 + b\, m + c)\, e^{mx} = 0$

$$\Rightarrow a\, m^2 + b\, m + c = 0$$

The above equation is an algebraic equation which is known as the **Auxiliary Equation**.We are to solve this to find the values of m for which e^{mx} satisfies the differential eqn.

So, we have reduced the task of solving a differential equation to one of solving an algebraicequation.

Let the two roots of the auxiliary eqn. be m_1 and m_2. Then $y = \exp(m_1x)$ and $y = \exp(m_2x)$ are two solutions.

Theorem : if y_1, y_2, y_3 etc., are solutions of a **Homogeneous**, **Linear** differential equation,then $(Ay_1 + By_2 + Cy_3 + \cdots)$ is also a solution of the same eqn., where A, B, C, etc., are arbitrary constants.

Proof :

$$A \times [a\, d^2y_1/dx^2 + b\, dy_1/dx + c\, y_1] = 0$$

$$B \times [a\, d^2y_2/dx^2 + b\, dy_2/dx + c\, y_2] = 0$$

Adding : $a\, d^2/dx^2 (Ay_1 + By_2) + b\, d/dx\, (Ay_1 + By_2) + c\, (Ay_1 + By_2) = 0$

Theorem : if y_1, y_2 are two linearly independent* solutions of a **2nd order**, **Homogeneous**, **Linear** differential equation, then $(Ay_1 + By_2)$ is the general solution of the differential equation, where A and B are arbitrary constants.

***Linear independence means : $(Ay_1 + By_2) = 0$ guarantees A = 0, B = 0.**

<u>Case I ($b^2 > 4ac$) : Here</u> m_1 and m_2 are two real and distinct roots of the auxiliaryeqn. The roots are unequal, which means that $\exp(m_1x)$ and $\exp(m_2x)$ are linearly independent solutions.

Hence, $\mathbf{y = A \exp(m_1x) + B \exp(m_2x)}$ is the general solution.

<u>Case II ($b^2 < 4ac$) :</u> Here m_1 and m_2 are complex and of the form : $m_1 = (\alpha + i\beta)$, $m_2 = (\alpha - i\beta)$. These are also unequal, hence $\exp(m_1x)$ and $\exp(m_2x)$ are linearly independent

solutions. So the general solution is of the form :

$$y = Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x} = e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x})$$
$$= e^{\alpha x}[\,A\{\cos\beta x + i\sin\beta x\} + B\{\cos\beta x - i\sin\beta x\}]$$
$$= e^{\alpha x}[\,C\cos\beta x + D\sin\beta x]$$

where $C = (A + B)$ and $D = i(A - B)$ are another set of arbitrary constants.

Case III ($b^2 = 4ac$) : Here the roots are equal, $m_1 = m_2 = m$ (say), where $m = (-b/2a)$.
Now the two solutions $\exp(m_1x)$ and $\exp(m_2x)$ are identical and cannot be combined to form the general solution.

$$y = Ae^{mx} + Be^{mx} = (A + B)\,e^{mx} = Ce^{mx},$$

where C is just another constant.

To obtain another linearly independent solution, try :

$$\mathbf{y = e^{mx} f(x)}$$

$$\Rightarrow dy/dx = m\,e^{mx} f(x) + e^{mx} f'(x)$$
$$\Rightarrow d^2y/dx^2 = m^2 e^{mx} f(x) + 2m\,e^{mx} f'(x) + e^{mx} f''(x)$$

Substituting in the differential equation (1) :

$$a\,e^{mx}[m^2 f(x) + 2mf'(x) + f''(x)] + b\,e^{mx}[m\,f(x) + f'(x)] + c\,e^{mx} f(x) = 0$$
$$\Rightarrow a[m^2 f(x) + 2mf'(x) + f''(x)] + b[m\,f(x) + f'(x)] + c\,f(x) = 0,$$

but $am^2 + bm + c = 0$, since m is the solution of the auxiliary equation
and also $2am + b = 0$, since, when $b^2 = 4ac$, $m = (-b/2a)$.

$$\text{Hence, } f''(x) = 0$$
$$\Rightarrow \mathbf{f(x) = Ax + C}$$

Thus, **$y = e^{mx} f(x) = e^{mx}(Ax + C)$ is the general solution.**
[The solution Ce^{mx} obtained at first, is absorbed in the second term above.]

Wronskian

If $y_1(x)$ and $y_2(x)$ are two solutions of a Linear, homogeneous differential equation, then the solutions are called **linearly dependent**, if there exists a non-trivial (not all zero) pair of coefficients c_1 and c_2, such that :

$$c_1 y_1 + c_2 y_2 = 0$$

Differentiating, we get :

$$c_1 y_1' + c_2 y_2' = 0$$

We can write this pair of equations in the matrix form as :

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution for c_1 and c_2, we must have : $\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = 0$,

i.e., $\mathbf{(y_1 y_2' - y_2 y_1') = 0.}$

If, on the other hand, the determinant is non-zero, then the only choice left is : $c_1 = c_2 = 0$
y_1, y_2 are called **linearly independent.** The determinant mentioned above is called the **Wronskian W(x).** The concept may easily be generalized to 'n' number of solutions.

Let us consider the differential equation :

$$\mathbf{a(x)\,d^2y/dx^2 + b(x)\,dy/dx + c(x)\,y = 0,}$$

which has two solutions : $y_1(x)$ and $y_2(x)$. We must have :

$$a(x)\,y_1'' + b(x)\,y_1' + c(x)\,y_1 = 0$$
$$\text{and } a(x)\,y_2'' + b(x)\,y_2' + c(x)\,y_2 = 0$$

Multiplying the first equation by y_2 and the second equation by y_1 and subtracting, we obtain :

$$a(x)\,(y_1''y_2 - y_2''y_1) + b(x)\,(y_1'y_2 - y_2'y_1) = 0$$

$$\text{i.e., } a(x)W'(x) + b(x)W(x) = 0$$

$$\Rightarrow W'(x)/W(x) = -b(x)/a(x)$$

$$\Rightarrow \int dW/W = -\int b(x)/a(x)\,dx$$

$$\Rightarrow \mathbf{W(x) = A\exp\left[-\int b(x)/a(x)\,dx\right]} \text{ ---- } (1)$$

To find the second solution :

If one of the solutions $y_1(x)$ of a second order, homogeneous, linear differential equation is known, then a second solution $y_2(x)$ may be found with the help of the Wronskian function.

The Wronskian W(x) of these two function, by definition, is : $y_1 y_2' - y_2 y_1'$

We view it as a differential equation for $y_2(x)$:

$$y_1(x)\;dy_2/dx - y_1'(x)\,y_2 = W(x)$$

$$\Rightarrow dy_2/dx - (y_1'/y_1)\,y_2 = W/y_1$$

This is clearly, a first order, linear, inhomogeneous differential equation in y_2.

The Integrating factor is : $\exp\left(-\int y_1'/y_1\,dx\right) = \exp\left(-\ln y_1\right) = 1/y_1$

Multiplying by this factor, we get :

$$(1/y_1)\,dy_2/dx - (y_1'/y_1^2)\,y_2 = W/y_1^2$$

$$\Rightarrow d/dx\,(y_2/y_1) = W/y_1^2$$

$$\Rightarrow (y_2/y_1) = \int (W/y_1^2)\,dx$$

$$\Rightarrow \mathbf{y_2(x) = y_1(x)\int (W/y_1^2)\,dx} \text{ ---- } (2)$$

Example :

Consider the differential equation : $d^2y/dx^2 - 2b\,dy/dx + b^2 y = 0$.

The trial $y = e^{mx} \Rightarrow m = b$. So we easily find a solution : $y_1(x) = e^{bx}$

The Wronskian for this differential equation is : $W(x) = A\exp\left[+\int 2b\,dx\right] = A\,e^{2bx}$ [by (1)]

Hence, a second solution is : $e^{bx}\int Ae^{2bx}/e^{2bx}$ [by (2)] $=$ $\mathbf{e^{bx} \times Ax}$, as we found earlier.

Second Order, In-homogeneous, Linear Differential Equation with Constant Coefficients

The differential eqn. is of the form :

$$\mathbf{a\,d^2y/dx^2 + b\,dy/dx + c\,y = f(x),} \text{ ---- } (1)$$

where f(x) is sometimes called the **'source term'**. Our task will be to find a general solution of this eqn.

Theorem : if $y_0(x)$ is the general solution of the **homogeneous** equation corresponding to (1) and $y_1(x)$ is any one particular solution of the full, inhomogeneous eqn. (1), then $y_0(x) + y_1(x)$ will be **the general solution of the of the full, inhomogeneous eqn.** (1).

We call $y_0(x)$ the **complementary function (C.F.)** and $y_1(x)$ the **Particular Integral function (P.I.).** The C.F. part will naturally involve two arbitrary constants. We have already learnt to find out the solution of a homogeneous eqn. So, we focus on finding the P.I.

We denote the operator 'd/dx' by 'D', so that our eqn. (1) may be written as :

$$[a\,D^2 + b\,D + c]\,y = f(x)$$

$$\Rightarrow y = \{1/[a\,D^2 + b\,D + c]\}\,f(x), \text{ or, } [a\,D^2 + b\,D + c]^{-1} f(x)$$

This is of course, a formal way of writing, because $[a\,D^2 + b\,D + c]$ is an operator and not a number. Actually, $[a\,D^2 + b\,D + c]^{-1}$ means the inverse of the operator.

1) $f(x) = e^{px}$

Note that $D\, e^{px} = d/dx\,(e^{px}) = p\, e^{px}$

$\Rightarrow\ D^2 e^{px} = p^2 e^{px}$, etc.,

which shows that while acting on e^{px}, the operator 'D' may be simply replaced by p

If **A** f(x) = λ f(x), where **A** is an operator, but λ is a number, then f(x) is called the '**eigen function**' of **A**, corresponding to the '**eigenvalue'** λ.

Thus, $\mathbf{P.I. = 1/\,[aD^2 + bD + c]\; e^{px} = 1/\,[ap^2 + bp + c]\; e^{px}}$

2) f(x) = sin px, or, cos px

$D \sin px = p \cos px \Rightarrow D^2 \sin px = (-p^2) \sin px,$

$D \cos px = -\ p \sin px \Rightarrow D^2 \cos px = (-p^2) \cos px,$

which means, that sin px, or, cos px are not eigen functions of D, but they are eigen functions of D^2. If our differential eqn. involves only even powers of D, i.e., D^2, D^4. etc., we can replace D^2 by $(-p^2)$. For example, if

$\mathbf{a\, d^2y/dx^2 + c\, y = \sin px,}$

then **P.I.** $\mathbf{= 1/\,[aD^2 + c] \sin px = 1/\,[-ap^2 + c] \sin px}$

If however odd powers of 'D' are involved, we have to '**rationalize**'. For example, if

$\mathbf{a\, d^2y/dx^2 + b\, dy/dx + c\, y = \cos px,}$

then $1/\,[aD^2 + bD + c]$ is written as : $[(aD^2 + c) - bD] / [(aD^2 + c) + bD]\, [(aD^2 + c) - bD]$

$= [(aD^2 + c) - bD]/\,[(aD^2 + c)^2 - (bD)^2]$

Hence **P.I.** $= [(aD^2 + c) - bD]/\,[(aD^2 + c)^2 - (bD)^2] \cos px$

$= [(aD^2 + c) - bD] \cos px/\,[(-ap^2 + c)^2 + b^2p^2]$

$\mathbf{= [(-ap^2 + c) \cos px - bp \sin px] / [(-ap^2 + c)^2 + b^2p^2]}$

<u>Example :</u>

$\mathbf{d^2y/dx^2 + 2\, dy/dx + 3\, y = \cos 2x}$

$\Rightarrow$ P.I. $= 1/\,[D^2 + 2D + 3] \cos 2x$

$= [(D^2 + 3) - 2D]/\,[(D^2 + 3)^2 - (2D)^2] \cos 2x$

$= [(D^2 + 3) - 2D] \cos 2x/\,[(-4 + 3)^2 + 16]$

$= [(-4 + 3) \cos 2x + 4 \sin 2x] / 17$

$\mathbf{= [(-4 + 3) \cos 2x + 4 \sin 2x] / 17.}$

3) f(x) = A + Bx + Cx² + . . . [A polynomial in x]

$1/(a\, D^2 + b\, D + c)$ may be written as :

$1/[c\, (aD^2/c + bD/c + 1)]$

$= 1/c \times 1/[1 + (bD/c + aD^2/c)]$

$= 1/c \times [1 + (bD/c + aD^2/c)]^{-1}$

$= 1/c\ \ [1 - (bD/c + aD^2/c) + (bD/c + aD^2/c)^2 - \cdots]$

[Since, $(1 + x)^{-1} = 1 - x + x^2 - \cdots$]

So, $1/[a\, D^2 + b\, D + c]\, (A + Bx + Cx^2 + \ldots)$

$= 1/c\ \ [1 - (bD/c + aD^2/c) + (bD/c + aD^2/c)^2 - \cdots]\, (a + bx + cx^2 + \ldots)$

Now, powers of D (i.e., d/dx) will act on powers of x and the result will vanish whenever the power of 'D' is higher. So, the series will eventually terminate. Let us clarify the procedure with an example :

[If you feel the above steps to be complicated, skip them and look at the example below]

<u>Example :</u>

$\mathbf{d^2y/dx^2 + 2\, dy/dx + 3\, y = 1 + 2x}$

$1/\,[D^2 + 2D + 3]\, (1 + 2x)$

$= 1/[3\,(D^2/3 + 2D/3 + 1)]$

$= 1/3 \times 1/[1 + (2D/3 + D^2/3)]$

$= 1/3 \times [1 + (2D/3 + D^2/3)]^{-1}$

$= 1/3\ [1 - (2D/3 + D^2/3) + (2D/3 + D^2/3)^2 - \cdots]$

$\Rightarrow$ **P.I.** $= 1/3\ [1 - (2D/3 + D^2/3) + (2D/3 + D^2/3)^2 - \cdots]\,(1 + 2x)$

$= 1/3\ [1 - 2D/3]\,(1 + 2x)$ [**Since the higher powers of D kills the function**]

$= 1/3\,(1 + 2x) - (2D/9)\,(2x) =$ $\mathbf{1/3\,(1 + 2x) - 4/9 = 2x/3 - 1/9}$.

4) **f(x) = $\mathbf{e^{px}\,V(x)}$**

$D\,f(x) = pe^{px}\,V(x) + e^{px}\,DV(x) = e^{px}\,(p + D)\,V(x)$

$D^2\,f(x) = \{p^2e^{px}\,V(x) + pe^{px}\,DV(x)\} + \{pe^{px}\,DV(x) + e^{px}\,D^2V(x)\}$

$= e^{px}\,\{p^2\,V(x) + 2p\,DV(x) + D^2V(x)\}$

$= e^{px}\,(p + D)^2\,V(x)$

. . . .

$\mathbf{D^n\,f(x) = e^{px}\,(p + D)^n\,V(x)}$

We can apply this rule in problems like : $\mathbf{d^2y/dx^2 + 2\,dy/dx + 3\,y = e^x\,x^2}$

However, the above rule has interesting applications in some special cases of type - 1) and type - 2) problems

<u>Example of type 1:</u>

$d^2y/dx^2 - 5\,dy/dx + 6\,y = e^{2x}$

$y = 1/(D^2 - 5\,D + 6)\ \ e^{2x} = 1/(D - 3)(D - 2)\ \ e^{2x}$

Following the standard prescription, you cannot substitute ‘2’ for ‘D’.

Re-write : $d^2y/dx^2 - 5\,dy/dx + 6\,y = \mathbf{e^{2x} \times 1}$

$(D^2 - 5\,D + 6)\,y = e^{2x}\,1$

$\Rightarrow\ y = 1/\,(D^2 - 5D + 6)\ \ e^{2x}\,1$

$= e^{2x}\ 1/[(D+2)^2 - 5\,(D+2) + 6]\ \ 1$ ---- [by rule (4)]

$= e^{2x}\ 1/[(D^2 + 4D + 4) - 5D - 10 + 6]\ \ 1$

$= e^{2x}\ 1/[D^2 - D]\ \ 1$

$= e^{2x}\ 1/[(D - 1)\,D]\ \ 1 = e^{2x}\ 1/[D - 1]\ \ x$ ---- [since $1/D$ is nothing but $\int dx$]

$= -e^{2x}\ 1/[1 - D]\ \ x = -e^{2x}\,[1 - D]^{-1}\,x = -e^{2x}\,[1 + D + D^2 + \ldots]\,x$

$=$ $\mathbf{-e^{2x}\,[x + 1]}$

<u>Example of type 2:</u>

$d^2y/dt^2 + \omega^2\,y = \sin \omega t$

[This is the forced harmonic oscillator without damping at resonance freq.]

$y = 1/[D^2 + \omega^2]\ \ \sin \omega t$, [Here D = stands for d/dt]

You cannot replace D^2 by $(-\,\omega^2)$. Following the standard prescription.

Now $(\sin \omega t)$ is the imaginary part of $(e^{i\omega t})$. So, write :

$y = \mathbf{Im}\,\{1/[D^2 + \omega^2]\ e^{i\omega t}\}$

You still cannot replace D by $(i\omega)$. So apply technique 4). Write :

$y = \mathbf{Im}\,\{1/[D^2 + \omega^2]\ e^{i\omega t} \times 1\}$

$= \mathbf{Im}\,\{e^{i\omega t} \times 1/[(D + i\omega)^2 + \omega^2]\ 1\}$

$= \mathbf{Im}\,\{e^{i\omega t} \times 1/[D^2 + 2i\omega D]\ 1\}$

$= \mathbf{Im}\,\{e^{i\omega t} \times 1/[D + 2i\omega]\ t\}$ ---- [since, $[1/D]\ 1 = t$]

$= \mathbf{Im}\,\{e^{i\omega t}/2i\omega \times 1/[1 + D/2i\omega]\ t\}$

$= \mathbf{Im}\left\{ e^{i\omega t}/2i\omega \times [1 - D/2i\omega + \cdots]\, t \right\}$

$= \mathbf{Im}\left\{ e^{i\omega t}/2i\omega \times [t - 1/2i\omega] \right\} = \mathbf{Im}\left\{ e^{i\omega t} \times [t/2i\omega + 1/4\omega^2] \right\}$

$= \mathbf{Im}\left\{ (\cos \omega t + i \sin \omega t)\,(t/2i\omega + 1/4\omega^2) \right\} = \boldsymbol{(\sin \omega t / 4\omega^2) - (t \cos \omega t / 2\omega)}$

Substituting in the diff. eqn. :

$$a\,[m^2 e^{mx} f(x) + 2m\, e^{mx} f'(x) + e^{mx} f''(x)]$$

$$+ b\,[m\, e^{mx} f(x) + e^{mx} f'(x)]$$

$$+ c\, e^{mx} f(x) = 0$$

$\Rightarrow e^{mx} f''(x)\,[am^2 + bm + c] + e^{mx} f'(x)\,[2am + b] + c\, e^{mx} f(x) = 0$

Note that : $m = (-\, b/2a) \Rightarrow [2am + b] = 0$, also $[am^2 + bm + c] = 0$

$\Rightarrow f''(x) = 0 \Rightarrow f'(x) = C \Rightarrow f(x) = Cx + D$

Thus, $y =$ **e^{mx} (Cx + D) is the general solution.**

Note that the general form includes the solution e^{mx}.