Laplace's Eqn. in Spherical Polar Co-ordinates

Laplace's Eqn. : $\nabla^2 V = 0$ in spherical polar co-ordinates reads :

$1/r^2\ \partial/\partial r\ (r^2\ \partial V/\partial r) + 1/(r^2 \sin\theta)\ \partial/\partial\theta\ (\sin\theta\ \partial V/\partial\theta) + 1/(r^2\sin^2\theta)\ \partial^2 V/\partial\phi^2 = 0$

We shall restrict ourselves, to problems with axial (azimuthal) symmetry, i.e., assume V is ϕ-indep.

$\Rightarrow$ $\mathbf{1/r^2\ \partial/\partial r\ (r^2\ \partial V/\partial r) + 1/(r^2\sin\theta)\ \partial/\partial\theta\ (\sin\theta\ \partial V/\partial\theta) = 0}$

Assume as before, $V(r, \theta) = R(r)\ f(\theta)$

$\Rightarrow$ $f/r^2\ d/dr\ (r^2\ dR/dr) + R/(r^2\sin\theta)\ d/d\theta\ (\sin\theta\ df/d\theta) = 0$

Multiply both sides by r^2 and divide by $R(r)\ f(\theta)$:

$\Rightarrow$ $(1/R)\ d/dr\ (r^2\ dR/dr) + (1/f\sin\theta)\ d/d\theta\ (\sin\theta\ df/d\theta) = 0$

The 2nd term is purely a function of 'θ' while 1st term is purely a function of 'r', which is possible only if both are equal to some const., say 'λ'.

$\Rightarrow$ $(1/f\sin\theta)\ d/d\theta\ (\sin\theta\ df/d\theta) = \lambda$ ––––– (1)

$\Rightarrow$ $(1/R)\ d/dr\ (r^2\ dR/dr) = -\lambda$ ––––– (2)

Eqn.(1) $\Rightarrow$ $\mathbf{(1/\sin\theta)\ d/d\theta\ (\sin\theta\ df/d\theta) = \lambda f}$

Put $\mathbf{x = \cos\theta}$ $\Rightarrow$ $df/d\theta = (df/dx)\ (dx/d\theta) = (df/dx)\ (-\sin\theta)$

$\Rightarrow$ $(\sin\theta\ df/d\theta) = (df/dx)(-\sin^2\theta) = (df/dx)(x^2 - 1)$

$\Rightarrow$ $d/d\theta\ (\sin\theta\ df/d\theta) = d/dx\ \{(x^2 - 1)\ df/dx\}\ dx/d\theta$

$= d/dx\ \{(x^2 - 1)\ df/dx\}(-\sin\theta)$

$\Rightarrow$ $(1/\sin\theta)\ d/d\theta\ (\sin\theta\ df/d\theta) = d/dx\ \{(1 - x^2)\ df/dx\}$

Thus, eqn.(1) $\Rightarrow$ $\mathbf{d/dx\ \{(1 - x^2)\ df/dx\} - \lambda f = 0}$,

$\Rightarrow$ $\mathbf{(1 - x^2)\ d^2f/dx^2 - 2x\ df/dx - \lambda f = 0}$, which is nothing but the **Legendre eqn.**

We know, that if we wish to have a convergent solution for $x = \pm 1$, i.e., $\theta = 0$ and π, which we do wish, λ must be of the form : $-\ell(\ell + 1)$.

The corresponding solution is written as : $P_\ell(x) = P_\ell(\cos\theta)$

Eqn.(2) $\Rightarrow$ $d/dr\ (r^2\ dR/dr) = -\lambda R = \ell(\ell + 1)\ R$

Try a solution of the form : $\mathbf{R = r^n}$ $\Rightarrow$ $dR/dr = n\ r^{n-1}$

$\Rightarrow$ $r^2\ dR/dr = n\ r^{n+1}$

$\Rightarrow$ $d/dr\ (r^2\ dR/dr) = n\ (n + 1)\ r^n = \ell(\ell + 1)\ r^n$

$\Rightarrow$ $n\ (n + 1) = \ell(\ell + 1)$

$\Rightarrow$ $n^2 + \mathbf{n} - \ell(\ell + 1) = 0$

$\Rightarrow$ $n^2 + \mathbf{n\ (\ell + 1) - n\ \ell} - \ell(\ell + 1) = 0$

$\Rightarrow$ $\mathbf{\{n + (\ell + 1)\}\ \{n - \ell\}} = 0$ $\Rightarrow$ $\mathbf{n = \ell}$, **or,** $\mathbf{n = -(\ell + 1)}$

i.e., $\mathbf{R = Ar^{\ell} + B/r^{\ell + 1}}$

Thus, the general solution is : $\mathbf{V(r, \theta) = \Sigma\ (A_\ell\ r^{\ell} + B_\ell / r^{\ell + 1})\ P_\ell(\cos\theta)}$

Example : A grounded conducting sphere, placed in a uniform electric field

For a uniform electric field $\mathbf{E} = E_0\ \mathbf{k}$ $\Rightarrow$ $V = -E_0\ z = -E_0\ r\cos\theta + C$

i) For $r \to \infty$, $V = -E_0 r \cos\theta + C$

ii) For $r = a$, $V = 0$ (since the conducting sphere is earthed)

For $r \to \infty$, $V(r, \theta) = \boldsymbol{\Sigma} (A_\ell r^\ell) P_\ell(\cos\theta) = A_0 + A_1 r \cos\theta + \cdots = -E_0 r \cos\theta + C$

$\Rightarrow$ $A_0 = C$, $A_1 = -E_0$, and $A_\ell = 0$ for all other ℓ.

$\Rightarrow$ $V(r, \theta) = C - E_0 r \cos\theta + \Sigma B_\ell / r^{\ell+1} P_\ell(\cos\theta)$

At $r = a$: $V(r, \theta) = C - E_0 a \cos\theta + \Sigma B_\ell / a^{\ell+1} P_\ell(\cos\theta)$

$\Rightarrow$ $0 = C P_0(\cos\theta) - E_0 a P_1(\cos\theta) + \Sigma B_\ell / a^{\ell+1} P_\ell(\cos\theta)$

We can compare term by term, because Legendre polynomials are all linearly independent.

$\Rightarrow$ $C = -B_0/a$, $E_0 a = B_1/a^2$ and $B_\ell = 0$ for all other 'ℓ'

$\Rightarrow$ $B_0 = -C a$, $B_1 = E_0 a^3$

$\Rightarrow$ $\mathbf{V(r, \theta) = C - E_0 r \cos\theta - C a/r + E_0 a^3/r^2 \cos\theta}$

$\mathbf{= C(1 - a/r) + E_0 (-r + a^3/r^2) \cos\theta.}$