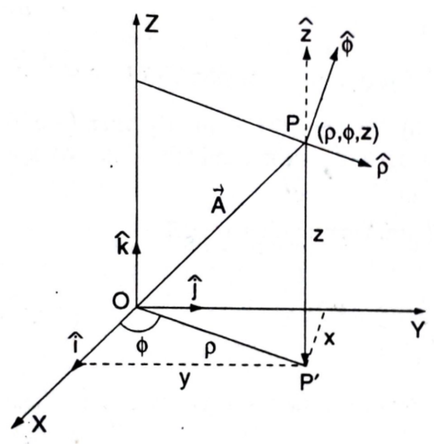
Velocity and acceleration of a particle in cylindrical polar coordinates:



In cylindrical polar coordinates the coordinates of a point P is represented by (ρ, ϕ, z). The position vector of P is given by

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = \rho Cos\phi\hat{i} + \rho Sin\phi\hat{j} + z\hat{k}$ ....(1)

$\frac{\partial \vec{r}}{\partial \rho}$ is a vector in the direction of increasing ρ, so that a unit vector in this direction is given by

$\hat{\rho} = \frac{\partial \vec{r}}{\partial \rho} / \left| \frac{\partial \vec{r}}{\partial \rho} \right|$ ....(2)

Now, $\frac{\partial \vec{r}}{\partial \rho} = Cos\phi\hat{i} + Sin\phi\hat{j}$ and $\left| \frac{\partial \vec{r}}{\partial \rho} \right| = \sqrt{Cos^2\phi + Sin^2\phi} = 1$.

$\therefore \hat{\rho} = Cos\phi\hat{i} + Sin\phi\hat{j}$ ....(3)

Similarly, unit vectors in the direction of increasing ϕ and increasing z are given by

$\hat{\phi} = \frac{\partial \vec{r}}{\partial \phi} / \left| \frac{\partial \vec{r}}{\partial \phi} \right|$ and $\hat{z} = \frac{\partial \vec{r}}{\partial z} / \left| \frac{\partial \vec{r}}{\partial z} \right|$ respectively.

Now, $\frac{\partial \vec{r}}{\partial \phi} = -\rho Sin\phi\hat{i} + \rho Cos\phi\hat{j}$ and $\left| \frac{\partial \vec{r}}{\partial \phi} \right| = \sqrt{(-\rho Sin\phi)^2 + (\rho Cos\phi)^2} = \rho$.

$\therefore \hat{\phi} = -Sin\phi\hat{i} + Cos\phi\hat{j}$ ....(4)

And $\frac{\partial \vec{r}}{\partial z} = \hat{k}$ and $\left| \frac{\partial \vec{r}}{\partial z} \right| = 1$.

$\therefore \hat{z} = \hat{k}$ ....(5)

Equations (3), (4) and (5) can be solved to get the expressions of unit vectors $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{\rho}, \hat{\phi}, \hat{z}$ and are given by,

$\hat{i} = Cos\phi\hat{\rho} - Sin\phi\hat{\phi}$ ....(6)

$\hat{j} = Sin\phi\hat{\rho} + Cos\phi\hat{\phi}$ ....(7)

$\hat{k} = \hat{z}$

Now, $\frac{d\hat{\rho}}{dt} = \frac{d\hat{\rho}}{d\phi}\frac{d\phi}{dt} = (-Sin\phi\hat{i} + Cos\phi\,\hat{j})\frac{d\phi}{dt} = \hat{\phi}\frac{d\phi}{dt}$ ....(8)

$\frac{d\hat{\phi}}{dt} = \frac{d\hat{\phi}}{d\phi}\frac{d\phi}{dt} = (-Cos\phi\hat{i} - Sin\phi\,\hat{j})\frac{d\phi}{dt} = -\hat{\rho}\frac{d\phi}{dt}$ ....(9)

And $\frac{d\,\hat{z}}{dt} = 0$ ....(10)

Position vector of point P can be written as,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= \rho Cos\phi(Cos\phi\hat{\rho} - Sin\phi\hat{\phi}) + \rho Sin\phi(Sin\phi\hat{\rho} + Cos\phi\hat{\phi}) + z\,\hat{z} \quad \text{[Using equations (6),(7)]}$$
$$= \rho\hat{\rho} + z\,\hat{z} \text{................(11)}$$

Velocity:

Velocity of any particle having the instantaneous position vector $\vec{r}$ is given by,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \frac{dz}{dt}\hat{z}$$
$$= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\hat{\phi}\frac{d\phi}{dt}\right) + \frac{dz}{dt}\hat{z}$$
$$= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi} + \frac{dz}{dt}\hat{z}$$
$$= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\,\hat{z} \text{...................(12)}$$

Acceleration:

Acceleration of the particle is given by,

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{d\rho}{dt}\hat{\rho}+\rho\frac{d\phi}{dt}\hat{\phi}+\frac{dz}{dt}\hat{z}\right)$$

$$=\frac{d^2\rho}{dt^2}\hat{\rho}+\frac{d\rho}{dt}\frac{d\hat{\rho}}{dt}+\frac{d\rho}{dt}\frac{d\phi}{dt}\hat{\phi}+\rho\frac{d^2\phi}{dt^2}\hat{\phi}+\rho\frac{d\phi}{dt}\frac{d\hat{\phi}}{dt}+\frac{d^2z}{dt^2}\hat{z}$$

$$=\frac{d^2\rho}{dt^2}\hat{\rho}+\frac{d\rho}{dt}\left(\hat{\phi}\frac{d\phi}{dt}\right)+\frac{d\rho}{dt}\frac{d\phi}{dt}\hat{\phi}+\rho\frac{d^2\phi}{dt^2}\hat{\phi}+\rho\frac{d\phi}{dt}\left(-\hat{\rho}\frac{d\phi}{dt}\right)+\frac{d^2z}{dt^2}\hat{z}$$

$$=\left[\frac{d^2\rho}{dt^2}-\rho\left(\frac{d\phi}{dt}\right)^2\right]\hat{\rho}+\left(\rho\frac{d^2\phi}{dt^2}+2\frac{d\rho}{dt}\frac{d\phi}{dt}\right)\hat{\phi}+\frac{d^2z}{dt^2}\hat{z}$$

$$=\left(\ddot{\rho}-\rho\dot{\phi}^2\right)\hat{\rho}+\left(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}\right)\hat{\phi}+\ddot{z}\,\hat{z}\text{....................}(13)$$

Note: Any arbitrary vector $\vec{A}$, in cylindrical polar coordinates can be written as

$$\vec{A}=A_\rho\hat{\rho}+A_\phi\hat{\phi}+A_z\,\hat{z}\text{ ....}(14)$$

Where A_ρ, A_ϕ, A_z are the components of $\vec{A}$ in the directions of $\hat{\rho},\hat{\phi},\hat{z}$ respectively.

Now,

$$\frac{d\vec{A}}{dt}=\frac{dA_\rho}{dt}\hat{\rho}+A_\rho\frac{d\hat{\rho}}{dt}+\frac{dA_\phi}{dt}\hat{\phi}+A_\phi\frac{d\hat{\phi}}{dt}+\frac{dA_z}{dt}\hat{z}+A_z\frac{d\hat{z}}{dt}$$

$$=\frac{dA_\rho}{dt}\hat{\rho}+A_\rho\left(\hat{\phi}\frac{d\phi}{dt}\right)+\frac{dA_\phi}{dt}\hat{\phi}+A_\phi\left(-\hat{\rho}\frac{d\phi}{dt}\right)+\frac{dA_z}{dt}\hat{z}+A_z.0\qquad\text{[Using equations (8),(9),(10)]}$$

$$=\left(\frac{dA_\rho}{dt}-A_\phi\frac{d\phi}{dt}\right)\hat{\rho}+\left(\frac{dA_\phi}{dt}+A_\rho\frac{d\phi}{dt}\right)\hat{\phi}+\frac{dA_z}{dt}\hat{z}\text{..................}(15)$$

If we put $\vec{A}=\vec{r}$ and $A_\rho=\rho, A_\phi=0, A_z=z$, in eq. (15), we get velocity $\vec{v}=\frac{d\vec{r}}{dt}$.

Problem:

A force $\vec{F}$ is given in Cartesian coordinates as $\vec{F}=yB\hat{i}-xB\hat{j}$, where B is a constant. Find the component F_ϕ in cylindrical polar coordinates.

Solution:

We have,

$$\vec{F} = yB\hat{i} - xB\hat{j}$$
$$= B\rho Sin\phi\left(Cos\phi\hat{\rho} - Sin\phi\hat{\phi}\right) - B\rho Cos\phi\left(Sin\phi\hat{\rho} + Cos\phi\hat{\phi}\right)$$
$$= -B\rho\hat{\phi}$$

Thus, $F_\phi = -B\rho$.

Problem:

A particle of charge q and mass m moves in a uniform magnetic field $\vec{B} = B\hat{z}$ and an electric field $\vec{E} = \dfrac{a}{\rho}\hat{\rho}$, where the constants a and B may be either positive or negative. Set up the equation of motion in cylindrical polar coordinates and show that $m\rho^2\dot{\phi} + \dfrac{qB}{2c}\rho^2$ is a constant of motion. Assume that the force due to electric field is $q\vec{E}$ and that due to magnetic field is $\dfrac{q}{c}\left(\vec{v}\,\mathrm{x}\,\vec{B}\right)$.

Solution:

Equation of motion of the particle is given by,

$$m\vec{a} = \vec{F} = q\vec{E} + \frac{q}{c}\left(\vec{v}\,\mathrm{x}\,\vec{B}\right)$$

In cylindrical polar coordinates, velocity and acceleration are given by,

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\,\hat{z}, \vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi} + \ddot{z}\,\hat{z}$$

$$\therefore m\left[\left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi} + \ddot{z}\,\hat{z}\right] = q\frac{a}{\rho}\hat{\rho} + \frac{q}{c}\left[\left(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}\right)\mathrm{x}\,B\hat{z}\right]$$
$$= q\frac{a}{\rho}\hat{\rho} - \frac{q}{c}\dot{\rho}B\hat{\phi} + \frac{q}{c}\rho\dot{\phi}B\hat{\rho}$$
$$= \left(\frac{qa}{\rho} + \frac{q\dot{\phi}B\rho}{c}\right)\hat{\rho} - \frac{q\dot{\rho}B}{c}\hat{\phi}$$

The equations of motion are given by,

$$m\left(\ddot{\rho} - \rho\dot{\phi}^2\right) = \frac{qa}{\rho} + \frac{q\dot{\phi}B\rho}{c}$$
$$m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) = -\frac{q\dot{\rho}B}{c}$$
$$m\ddot{z} = 0$$

Multiplying the second equation by ρ , we have

$$m\left(\rho^2\ddot{\phi}+2\rho\dot{\rho}\dot{\phi}\right)=-\frac{q\rho\dot{\rho}B}{c}$$

$$Or,\frac{d}{dt}\left(m\rho^2\dot{\phi}+\frac{qB}{2c}\rho^2\right)=0$$

$\therefore m\rho^2\dot{\phi}+\frac{qB}{2c}\rho^2 =$ a constant of motion.