Maxima – Minima

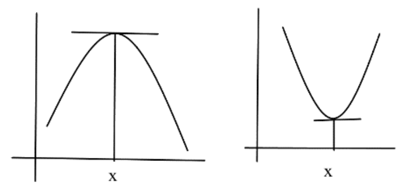
Function of a single variable

One dimensional Taylor Expansion :

If we know the values of a function f(x), along with its derivatives, at a point ‘x’, then Taylor expansion gives us the value of the function at a neighbouring point (x + h) :

$$\mathbf{f(x+h) = f(x) + h\,f'(x) + h^2/2!\,f''(x) + h^3/3!\,f'''(x)} + \cdots$$

If the function has a (local) maximum or minimum at a point ‘x’ then the value of the function should be locally stationary. i.e. f(x) should be equal to f(x + h) for a small ‘h’. This implies : **$f'(x) = 0$ up to 1st order in h.** Pictorially also, one can see that the tangent at the point ‘x’ is parallel to the x-axis.



Now, **up to 2nd order in h** :

$$f(x+h) = f(x) + h^2/2!\,f''(x) \quad [\text{since } f'(x) = 0]$$

If the point ‘x’ is a maximum, we require f(x + h) < f(x) for both +ve and -ve ‘h’. This implies : **$f''(x) < 0$.**

Similarly, if the point ‘x’ is a minimum, we require f(x + h) > f(x) for both +ve and -ve ‘h’, which implies : **$f''(x) > 0$.**

Thus, condition for **Maximum** is : **$f'(x) = 0,\ f''(x) < 0$**

and condition for **Minimum** is : **$f'(x) = 0,\ f''(x) > 0$**

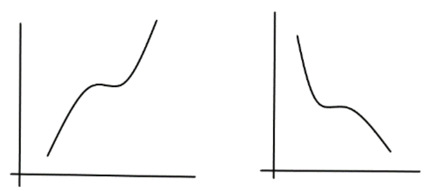
If, however, $f''(x) = 0$, we need to go to higher orders. **Up to 3rd order in h** :

$$\mathbf{f(x+h) = f(x) + h^3/3!\,f'''(x)} \quad [\text{since } f'(x) = 0,\ f''(x) = 0]$$

Now, we find that, if **$f'''(x) > 0$**, then f(x + h) > f(x) for +ve ‘h’
and f(x + h) < f(x) for -ve ‘h’

The situation gets reversed if **$f'''(x) < 0$**.

Thus **f(x) has neither a maximum nor a minimum** at x. It is called a ‘**Point of inflection’.**



f'''(x) > 0   f'''(x) < 0

The general prescription is thus as follows :

Keep on differentiating f(x) and evaluate the derivatives at ‘x’.

If the first non-zero derivative occurs at an odd order (like 3), (be it +ve or –ve) then ‘x’ is a **point of inflection**.

If the first non-zero derivative occurs at an even order (say n = 2), then there is **either a max., or a min.** at ‘x’.

If $f^n(x) < 0$, it’s a max., if $f^n(x) > 0$, it’s a min.

Examples :

1. **$f(x) = x^2$** $\Rightarrow f'(x) = 2x$. So, $f'(x) = 0$ at $x = 0$.
   $f''(x) = 2$ at $x = 0$ (actully, at all points) $\Rightarrow x = 0$ is a minimum point.
2. **$f(x) = 10 - x^2$** $\Rightarrow f'(x) = -2x$. So, $f'(x) = 0$ at $x = 0$.
   $f''(x) = -2$ at $x = 0$ (actully, at all points) $\Rightarrow x = 0$ is a point of maxima.
3. **$f(x) = x^3$** $\Rightarrow f'(x) = 3x^2$. So, $f'(x) = 0$ at $x = 0$.
   $f''(x) = 6x = 0$ at $x = 0 \Rightarrow f'''(x) = 6$ (non-zero at 3rd order) at $x = 0$
   $\Rightarrow x = 0$ is a point of inflection.

4. $\mathbf{f(x) = x^4}$ $\Rightarrow f'(x) = 4x^3$. So, $f'(x) = 0$ at $x = 0$.
$f''(x) = 12x^2 = 0$ at $x = 0$ $\Rightarrow$ $f'''(x) = 24x = 0$ at $x = 0$
$\Rightarrow$ $f''''(x) = 24$ (non-zero at 4th order) at $x = 0$, hence, $x = 0$ is a minimum point.

Function of a two variables

Two dimensional Taylor Expansion :

$$\mathbf{f(x+h, y+k) = f(x, y) + \{h f_x(x, y) + k f_y(x, y)\} + \{h^2 f_{xx}(x, y) + k^2 f_{yy}(x, y) + 2\, hk\, f_{xy}(x, y)\}/ 2! + \cdots}$$

In this case, if the function has a (local) maximum or minimum at a point (x, y) then the value of the function should be stationary within a small patch around the point (x, y).

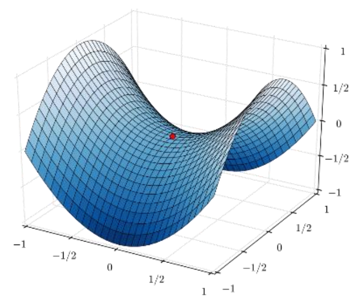
Upto 1st order :

$f(x+h, y+k) = f(x, y) \Rightarrow$ **$f_x(x, y) = 0$, $f_y(x, y) = 0$, if x and y are independent variables.**

To find whether there is a maximum or minimum at this point, we are to evaluate the **Hessian determinant** :

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = (f_{xx}\ f_{yy} - f_{xy}^2) = H.$$

If H > 0, and $f_{xx} > 0$, the point (x, y) is a **minimum point;**
if H > 0, and $f_{xx} < 0$, the point (x, y) is a **maximum point;**
if H = 0, no such conclusion can be drawn
and if H < 0, the point (x, y) is a **'saddle point'.**



A saddle point is one which is maximum along one axis and minimum along another. Typical example is the point at the middle of a horse' saddle (hence the name).

Constrained Maxima/Minima
Lagrange' s Method of Undetermined Multiplier

Suppose you area given a function of several variables f(x, y, z) and you are to find the maxima / minima of the function, subject to one or more '**constraints'** of the form $\phi(x, y, z)$ = const. (The number of variables – the number of constraints) is usually called the '**degrees of freedom'**. the Since at the extremum point, the function is locally stationary,

$$df(x,y) = 0 \Rightarrow f_x\, dx + f_y\, dy + f_z\, dz = 0 \text{ ---- (1)}$$

However, dx, dy, dz are not independent here, because of the constraint. So, we cannot equate their coefficient to zero.

$$\phi(x, y, z) = c \Rightarrow d\phi(x, y, z) = 0 \Rightarrow \phi_x\, dx + \phi_y\, dy + \phi_z\, dz = 0 \text{ ---- (2)}$$

Multiply eqn.(2) by some constant 'λ' (which is not determined so far) and add with eqn.(1) :

$$\Rightarrow \{f_x + \lambda\, \phi_x\}\, dx + \{f_y + \lambda\, \phi_y\}\, dy + \{f_z + \lambda\, \phi_z\}\, dz = 0 \text{ ---- (3)}$$

Since we have three variables and one constraints here, the number of independent variables = 2. We may choose, say as x and y as the independent ones. Now we adjust 'λ' to make the coefficient of dz = 0, i.e.,

$$f_z + \lambda\, \phi_z = 0 \text{ ---- (4)}$$

[which is possible by making $\lambda = -f_z / \phi_z$]

Now, eqn. (3) $\Rightarrow \{f_x + \lambda\, \phi_x\}\, dx + \{f_y + \lambda\, \phi_y\}\, dy = 0$,

where dx and dy are chosen as independent. **So, now we can equate their coefficient to zero.**

$\Rightarrow f_x + \lambda\, \phi_x = 0$ ---- (5)

$f_y + \lambda\, \phi_y = 0$ ---- (6)

Solving (4), (4) and (6), together with the constraint eqn. $\phi(x, y, z) = c$, we can find x, y, z and also λ. If we have two constraints, say $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$, then :

$d\phi(x, y, z) = 0 \Rightarrow \phi_x\, dx + \phi_y\, dy + \phi_z\, dz = 0$ ---- (7)

$d\psi(x, y, z) = 0 \Rightarrow \psi_x\, dx + \psi_y\, dy + \psi_z\, dz = 0$ ---- (8)

Multiplying eqn.(7) by 'λ' and (8) by 'μ' and adding with eqn.(1) :

$\Rightarrow \{f_x + \lambda\, \phi_x + \mu\, \psi_x\}\, dx + \{f_y + \lambda\, \phi_y + \mu\, \psi_y\}\, dy + \{f_z + \lambda\, \phi_z + \mu\, \psi_z\}\, dz = 0$ ---- (9)

Now, only one variable is independent say 'x'. We can adjust 'λ' and 'μ' to make :

$f_y + \lambda\, \phi_y + \mu\, \psi_y = 0$ ---- (10)

and $f_z + \lambda\, \phi_z + \mu\, \psi_z = 0$ ---- (11)

[which is possible by solving eqns. (10) and (11) for λ and μ]

Then, equating the coefficient of 'dx' to zero we get :

$f_x + \lambda\, \phi_x + \mu\, \psi_x = 0$ ---- (12)

So, we shall need to solve eqns (10), (11), (12), together with the two constraint equations : $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$, to find x, y, z , λ and μ.

Examples :

1. Maximize the area of a rectangle formed by a string of fixed length 'L'.

   Let the sides of the rectangle be x and y. Then, area A = xy and the perimeter = 2(x + y) = L.

   $\Rightarrow \partial A/\partial x + \lambda\, \partial L/\partial x = y + 2\lambda = 0$

   $\partial A/\partial y + \lambda\, \partial L/\partial y = x + 2\lambda = 0$

   $\Rightarrow x = y$. i.e. the rectangle must be a square.

   Substituting in the constraint eqn. : $4x = L \Rightarrow x = L/4$.

2. Maximize the volume of a rectangular parallelepiped, subject to the constraint that its total surface area is constant.

   Let the sides of the parallelepiped be x, y and z. Then, its volume V = xyz and the surface area S = 2(xy + yz + zx) = const.

   $\Rightarrow V_x + \lambda\, S_x = 0 \Rightarrow yz + 2\lambda\, (y + z) = 0$ ---- (a)

   $V_y + \lambda\, S_y = 0 \Rightarrow xz + 2\lambda\, (x + z) = 0$ ---- (b)

   $V_z + \lambda\, S_z = 0 \Rightarrow xy + 2\lambda\, (x + y) = 0$ ---- (c)

Multiply eqn. (a) by x and eqn.(b) by y and subtract :

$\Rightarrow 2\lambda\, (xz - yz) = 0 \Rightarrow x = z$

Similarly, we can show that y = z and z = x.

$\Rightarrow S = 2(xy + yz + zx) = 6x^2$

$\Rightarrow x = \sqrt{(S/6)}$, in terms of the given data.