Eigen value Problem

A square matrix ‘M’, acting on a column matrix (vector) ‘x’, usually changes both the magnitude and direction of ‘x’. If, in some special case, the action of ‘M’ only changes the magnitude of ‘x’ by a factor, say ‘λ’, we call ‘x’ to be an **eigen vector** of M, corresponding to the **eigen value λ** :

$$M\,x = \lambda\, x \quad \text{---- (1)}$$

$$\Rightarrow\; M\,x = \lambda\, I\, x, \text{ where I is the identity matrix}$$

$$\Rightarrow\; (M - \lambda I)\, x = 0, \text{ where RHS is a zero column matrix}$$

We seek a **non-zero** solution for ‘x’, because ‘x = 0’ satisfies the equation ‘$Mx = \lambda x$’, for all ‘M’ and all ‘λ’ trivially.

However, if the matrix $(M - \lambda I) = N$ has an inverse, then $Nx = 0 \Rightarrow N^{-1} Nx = 0 \Rightarrow x = 0$.

So, **‘N’ must not have an inverse.** That is possible only if

$$\det N = \det | M - \lambda I | = 0. \quad \text{----- (2)}$$

Eqn.(2) is called the ‘**characteristic or the secular eqn.’** and the determinant is called ‘**the secular determinant’.** If the matrix ‘M’ is $n \times n$, eqn.(2) is a n-th degree equation in ‘λ’ and it will have ‘n’ roots, which may not be all different. For example, the roots for a 3×3 matrix may be $\lambda = 1, 1, 2$.

The number of times a root of the chaarceristic eqn. is repeated, is called the ‘algebraic multiplicity’of that root.

The ‘algebraic multiplicity’ of the root ‘1’ in the above example is 2.

Substituting one of the values of λ in eqn.(1), we can find the corresponding eigen vector ‘x’. One may have more than one linearly independent eigen vectors for one ‘λ’.

The number of linearly independent eigen vectors for an eigen value ‘λ’ is called its ‘degeneracy’ or, ‘geometric multiplicity’.

Geometric multiplicity is always less than or equal to the algebraic multiplicity.

Example 1 :

Find the eigen values and normalized eigen vectors of the matrix : $\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The characteristic eqn. is : $\begin{vmatrix} 0-\lambda & -i & 0 \\ i & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0 \;\Rightarrow\; (\lambda^2 - 1)(-\lambda) = 0$

$\Rightarrow\;$ **$\lambda = 1,\ -1,\ 0$**

Eigen vectors :

For $\lambda = 1$, $\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \;\Rightarrow\; \begin{aligned} -i\,y_1 &= x_1 \quad \text{---- (a)} \\ i\,x_1 &= y_1 \quad \text{---- (b)} \\ 0 &= z_1 \quad \text{---- (c)} \end{aligned}$

Note that we cannot determine x_1, y_1, z_1 completely, but only find their ratio.

If we **choose** $x_1 = c$, both (a) and (b) $\Rightarrow$ $y_1 = ic$ and (c) $\Rightarrow$ $z_1 = 0$

Thus the eigen vector looks like : $\begin{bmatrix} \mathbf{c} \\ \mathbf{ic} \\ \mathbf{0} \end{bmatrix}$.

Since this is a complex vector, its norm (magnitude) is given by : $\mathbf{\{|c|^2 + |ic|^2 + 0\}^{1/2} = \sqrt{2}c}$.

Hence, the normalized eigen vector is : $\mathbf{1/\sqrt{2}} \begin{bmatrix} \mathbf{1} \\ \mathbf{i} \\ \mathbf{0} \end{bmatrix}$.

Similarly, for $\boldsymbol{\lambda = -1}$, we have : $-i\, y_2 = -x_2$ ---- (d)

$i\, x_2 = -y_2$ ---- (e)

$0 = -z_2$ ---- (f)

If we choose $x_2 = c'$, both (d) and (e) $\Rightarrow$ $y_2 = -ic'$ and (f) $\Rightarrow$ $z_2 = 0$.

Its norm (magnitude) is given by : $\mathbf{\{|c'|^2 + |-ic'|^2 + 0\}^{1/2} = \sqrt{2}c}$

Hence, the normalized eigen vector is : $\mathbf{1/\sqrt{2}} \begin{bmatrix} \mathbf{1} \\ \mathbf{-i} \\ \mathbf{0} \end{bmatrix}$.

Finally, for **for $\boldsymbol{\lambda = 0}$**, we have : $-i\, y_3 = 0$ ---- (g)

$i\, x_3 = 0$ ---- (h)

$0 = 0$ ---- (i)

Both (g) and (h) $\Rightarrow$ $x_3 = y_3 = 0$ and eqn.(i) is a 'blank' eqn., imposing **no restriction at all.**

If we choose $z_3 = c''$,

the norm of the vector is also c'' and the normalized eigen vector is : $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$.

<u>Similarity Transformation :</u>

The transformation of a matrix $\mathbf{M \to P^{-1} M P}$ is called a '**similarity transformation**'. Many property of the matrix M is maintained under such a transformation.

(i) det $|P^{-1} M P|$ = det $|P^{-1}|$ det $|M|$ det $|P|$
= det $|P^{-1}|$ det $|P|$ det $|M|$
= det $|P^{-1}P|$ det $|M|$
= det $|I|$ det $|M|$ = = det $|M|$

(ii) Tr $(P^{-1} M P)$ **[Trace of a matrix M is the sum of its diagonal elements : $\boldsymbol{\Sigma_i M_{ii}}$]**
$= \Sigma_i (P^{-1}M P)_{ii}$
$= \Sigma_{ijk} (P^{-1})_{ij} (M)_{jk} (P)_{ki}$
$= \Sigma_{ijk} (P)_{ki} (P^{-1})_{ij} (M)_{jk}$
$= \Sigma_k (P P^{-1}M)_{kk} = \Sigma_k (M)_{kk} = \text{Tr } M$

(iii) If $M x = \lambda x$, then $P^{-1} M x = \lambda P^{-1} x$
$\Rightarrow P^{-1} M (PP^{-1}) x = \lambda P^{-1} x$
$\Rightarrow P^{-1} M P (P^{-1} x) = \lambda (P^{-1} x)$

Thus, the eigen vector $x \to (P^{-1} x)$, while the eigen value remains the same.

Diagonalizing Matrix :

A matrix 'P', constructed by making the eigen vectors of 'M' stand side by side, is called the diagonalizing matrix for M. In the above example : $\mathbf{P} = \begin{bmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/\sqrt{2}} & \mathbf{0} \\ \mathbf{i/\sqrt{2}} & \mathbf{-i/\sqrt{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$.

is the diagonalizing matrix for $M = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

You may check that $P^{-1} M P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Types of Matrices

1) A matrix M is called ‘**symmetric**’ if : $\mathbf{M^T = M}$

2) A matrix M is called ‘**anti-symmetric**’ or ‘**skew-symmetric**’ if : $\mathbf{M^T = -M}$

3) A matrix M is called ‘**orthogonal**’ if : $\mathbf{M^T = M^{-1}}$, i.e., $\mathbf{M^T M = M M^T = I}$

4) A matrix M is called ‘**Hermitian**’ if : $\mathbf{M^\dagger = M}$

5) A matrix M is called ‘**anti-Hermitian**’ or ‘**skew-Hermitian**’ if : $\mathbf{M^\dagger = -M}$

6) A matrix M is called ‘**unitry**’ if : $\mathbf{M^\dagger = M^{-1}}$, i.e., $\mathbf{M^\dagger M = M M^\dagger = I}$

Theorems :

1. The eigen values of a unitary matrix satisfies the eqn. $|\lambda| = 1$.

Proof :

Let

$$M X = \lambda X, \quad \text{---- (1)}$$

where M is a unitary matrix with eigenvalue λ and the corresponding eigenvector X.
Taking the Hermitian adjoint of both sides :

$$X^\dagger M^\dagger = \lambda^* X^\dagger, \quad \text{---- (2)}$$

where * symbolizes complex conjugation. Multiplying :

$$(X^\dagger M^\dagger)(M X) = \lambda^*\lambda (X^\dagger X) = |\lambda|^2 (X^\dagger X),$$

but

$$(M^\dagger M) = I \Rightarrow (X^\dagger X) = |\lambda|^2 (X^\dagger X),$$

$$\Rightarrow (|\lambda|^2 - 1)(X^\dagger X) = 0$$

Now, $(X^\dagger X)$ cannot be zero unless $X = 0$

$$\Rightarrow |\lambda|^2 = 1, \text{ or, } \lambda \text{ is of the form } e^{i\theta}.$$

2. All eigenvalues of a Hermitian matrix are real.

Proof :

Let

$$M X = \lambda X, \quad \text{---- (1)}$$

where M is a Hermitian matrix with eigenvalue λ and the corresponding eigenvector X.
Taking the Hermitian adjoint of both sides :

$$X^\dagger M^\dagger = \lambda^* X^\dagger,$$

but, $M^\dagger = M \Rightarrow X^\dagger M = \lambda^* X^\dagger$, ---- (2)

Multiplying (1) with $X^\dagger$ from left :

$$X^\dagger M X = \lambda (X^\dagger X) \quad \text{---- (3)}$$

Multiplying (2) with X from right :

$$X^\dagger M X = \lambda^* (X^\dagger X) \quad \text{---- (4)}$$

Subtracting (3) from (4) :

$$0 = (\lambda^* - \lambda)(X^\dagger X)$$

Now, $(X^\dagger X)$ cannot be zero unless $X = 0$

$$\Rightarrow \lambda^* = \lambda, \text{ i.e., } \lambda \text{ is real.}$$

3. The eigenvalues of a skew-Hermitian matrix are purely imaginary, or zero.

Proof :

Let

$$M X = \lambda X, \text{ ---- (1)}$$

where M is a skew-Hermitian matrix with eigenvalue λ and the corresponding eigenvector X.

Taking the Hermitian adjoint of both sides :

$$X^\dagger M^\dagger = \lambda^* X^\dagger,$$

$$\text{but, } M^\dagger = -M \Rightarrow X^\dagger M = -\lambda^* X^\dagger, \text{ ---- (2)}$$

Multiplying (1) with $X^\dagger$ from left :

$$X^\dagger M X = \lambda (X^\dagger X) \text{ ---- (3)}$$

Multiplying (2) with X from right :

$$X^\dagger M X = -\lambda^* (X^\dagger X) \text{ ---- (4)}$$

Subtracting (3) from (4) :

$$0 = (\lambda^* + \lambda)(X^\dagger X)$$

Now, $(X^\dagger X)$ cannot be zero unless $X = 0$

$\Rightarrow \lambda^* = -\lambda$, i.e., λ is purely imaginary, or zero.

4. The eigen vectors of a Hermitian matrix corresponding to different eigen values are orthogonal.

Proof :

Let

$$M X_1 = \lambda_1 X_1, \text{ ---- (1)}$$
$$M X_2 = \lambda_2 X_2, \text{ ---- (2)}$$

where M is a Hermitian matrix, X_1 is an eigen vector, corresponding to the eigen value λ_1 and X_2 is an eigen vector, corresponding to the eigen value λ_2.

Taking the Hermitian adjoint of both sides of (2) :

$$X_2^\dagger M^\dagger = \lambda_2^* X_2^\dagger,$$

$$\text{but, } M^\dagger = M \text{ and } \lambda_2^* = \lambda_2$$

$$\Rightarrow X_2^\dagger M = \lambda_2 X_2^\dagger, \text{ ---- (3)}$$

Multiplying (1) with $X_2^\dagger$ from left :

$$X_2^\dagger M X_1 = \lambda_1 (X_2^\dagger X_1) \text{ ---- (4)}$$

Multiplying (3) with X_1 from right :

$$X_2^\dagger M X_1 = \lambda_2 (X_2^\dagger X_1) \text{ ---- (5)}$$

Subtracting (4) from (5) :

$$0 = (\lambda_2 - \lambda_1)(X_2^\dagger X_1)$$

Now, $\lambda_2 \neq \lambda_1 \Rightarrow (X_2^\dagger X_1) = 0$, i.e., X_1 and X_2 are orthogonal.

5. The eigen vectors of a skew-Hermitian matrix corresponding to different eigen values are also orthogonal.

Proof :

Let

$$M X_1 = \lambda_1 X_1, \text{ ---- (1)}$$
$$M X_2 = \lambda_2 X_2, \text{ ---- (2)}$$

where M is a skew-Hermitian matrix, X_1 is an eigen vector, corresponding to the eigen value λ_1 and X_2 is an eigen vector, corresponding to the eigen value λ_2.

Taking the Hermitian adjoint of both sides of (2) :

$$X_2^\dagger M^\dagger = \lambda_2^* X_2^\dagger,$$

$$\text{but, } M^\dagger = -M \text{ and } \lambda_2^* = -\lambda_2$$

$$\Rightarrow X_2^\dagger M = \lambda_2 X_2^\dagger, \text{ ---- (3)}$$

Multiplying (1) with $X_2^\dagger$ from left :

$$X_2^\dagger M X_1 = \lambda_1 (X_2^\dagger X_1) \text{ ---- (4)}$$

Multiplying (3) with X_1 from right :

$$X_2^\dagger M X_1 = \lambda_2 (X_2^\dagger X_1) \text{ ---- (5)}$$

Subtracting (4) from (5) :

$$0 = (\lambda_2 - \lambda_1)(X_2^\dagger X_1)$$

Now, $\lambda_2 \neq \lambda_1 \Rightarrow (X_2^\dagger X_1) = 0$, i.e., X_1 and X_2 are orthogonal.

1. The eigen vectors of a unitary matrix corresponding to different eigen values are also orthogonal.

Proof :

Let

$$M X_1 = \lambda_1 X_1, \text{ ---- (1)}$$

$$M X_2 = \lambda_2 X_2, \text{ ---- (2)}$$

where M is a unitary matrix, X_1 is an eigen vector, corresponding to the eigen value λ_1 and X_2 is an eigen vector, corresponding to the eigen value λ_2.

Taking the Hermitian adjoint of both sides of (2) :

$$X_2^\dagger M^\dagger = \lambda_2^* X_2^\dagger, \text{ ---- (3)}$$

where * symbolizes complex conjugation. Multiplying both sides of (3) and (1) :

$$(X_2^\dagger M^\dagger)(M X_1) = \lambda_2^* \lambda_1 (X_2^\dagger X_1)$$

but

$$(M^\dagger M) = I \Rightarrow (X_2^\dagger X_1) = \lambda_2^* \lambda_1 (X_2^\dagger X_1)$$

$$\Rightarrow (\lambda_2^* \lambda_1 - 1)(X_2^\dagger X_1) = 0$$

Now, if $\lambda_1 = \exp(i\theta_1)$ and $\lambda_2 = \exp(i\theta_2)$, where $\theta_1 \neq \theta_2$, then

$$\lambda_2^* \lambda_1 \neq 1$$

$$\Rightarrow (X_2^\dagger X_1) = 0, \text{ i.e., } X_1 \text{ and } X_2 \text{ are orthogonal}$$