Group velocity and phase velocity:

Phase velocity:

Phase velocity of a monochromatic wave is the velocity with which a crest or trough of the wave is propagated in a medium.

For example, let us consider a plane wave propagating in the positive direction of X-axis. If the wave is monochromatic, i.e. it has only one frequency component, it will be simple harmonic in nature and the wave function will be

$$y = ASin(\omega t - kx)$$

Where A is the amplitude of the wave and its phase at a point x at the instant of time t is $s = \omega t - kx$. If the phase remains constant with time, we must have

$$\frac{ds}{dt} = \omega - k\frac{dx}{dt} = 0$$

$$\therefore \frac{dx}{dt} = \frac{\omega}{k}$$

Therefore, the phase velocity of the wave is the velocity with which the constant phase surface moves (wavefront) moves in space.

$$\therefore u_{ph} = \frac{dx}{dt} = \frac{\omega}{k}$$

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Wave group:

A simple harmonic wave train as represented by the equation $y = ASin(\omega t - kx)$ is an idealised concept because it sets no limit on x and requires an infinitely long train of waves. No source of waves vibrates indefinitely. The vibrations die out due to interruption or loss of energy. In such a case, a train of waves of finite length is produced, which is called a wave train. When wave trains of different but close together frequencies, moving in the same direction, are superposed, the resultant pattern will be called a wave group.

A short enough wave train may be treated as a wave group, as it may be considered to be formed by the superposition of a theoretically infinite number of plane harmonic waves whose frequencies differ continuously. The frequencies are confined within a range depending on the length of the wave train. The shorter the length, the wider will be the effective frequency range.

The properties of a wave group differ greatly from those of a simple harmonic (hence monochromatic) wave train. When a wave group is made to pass through a wavelength analyser, it is found that the wave trains in the group do not have the same wavelength; instead there will be a finite, however, small spread up wavelength $\Delta\lambda$ about a mean wavelength λ_0 that corresponds to maximum energy (see figure). Other wavelengths in the range also carry energy, but the energy falls off as the wavelength differs more and more from λ_0. The relationship between the spreading wavelength with λ and the number of individual waves (N) is $\Delta\lambda = \frac{\lambda}{N}$. This implies that for higher the number of wavelength comprising the wave group the spread will be lesser about λ_0. When N is very large, i.e. $N \to \infty$, we get a well defined λ which corresponds to a simple harmonic wave train (monochromatic).

Mathematical representation of a wave group is rather complex. It may be looked upon as being synthesized by adding up an infinite number of wave trains whose frequencies differ by infinitesimal amounts. Besides, their amplitudes must be such as to produce the wave form of the wave concerned.

Group velocity:

Let us consider a wave group formed by the superposition of an infinite number of plane simple harmonic waves. If the medium through which the wave group travels be dispersive, i.e. the wave velocity depends on frequency, the maximum of the wave group travels with a velocity different from that of the component waves. This is called the group velocity. Energy travels with the group and has the velocity of the group.

Relation between the group velocity and the phase velocity:

Let us consider a simple case in which the group is formed by the superposition of two waves of equal amplitude but of slightly different frequency and wavelength moving in the same directions. The component waves are given by

$$y_1 = aSin(\omega t - kx),$$

$$y_2 = aSin\{(\omega + d\omega)t - (k + dk)x)\}$$

The resultant wave is given by

$$\begin{aligned} y = y_1 + y_2 &= aSin(\omega t - kx) + aSin\{(\omega + d\omega)t - (k + dk)x)\} \\ &= 2aSin\frac{1}{2}\{(\omega + d\omega)t - (k + dk)x) + (\omega t - kx)\} \text{ x } Cos\frac{1}{2}\{(\omega + d\omega)t - (k + dk)x) - (\omega t - kx)\} \\ &= 2aCos(\frac{1}{2}d\omega t - \frac{1}{2}dkx)Sin\{(\omega + \frac{1}{2}d\omega)t - (k + \frac{1}{2}dk)x\} \end{aligned}$$

From the sine term, we can see that the resultant wave has an angular frequency $(\omega + \frac{1}{2}d\omega)$ and a wave length constant $(k + \frac{1}{2}dk)$, both of which are the mean values for the component waves. Its phase velocity is $c = \frac{\omega + \frac{1}{2}d\omega}{k + \frac{1}{2}dk} = \frac{\overline{\omega}}{\overline{k}} \simeq \frac{\omega}{k}$, where $\overline{\omega}$ and $\overline{k}$ are the mean frequency and mean wave constant respectively. This is practically equal to $\frac{\omega}{k}$, i.e., the phase velocity of the either component wave. Since ω and k are very large compared to $d\omega$ and dk, the cosine term varies very slowly compared to the sine term and it may be taken as the amplitude of the resultant wave.

The group velocity of the wave is determined by the speed with which the crest and trough of the wave group (i.e. of the envelope) travels in space.

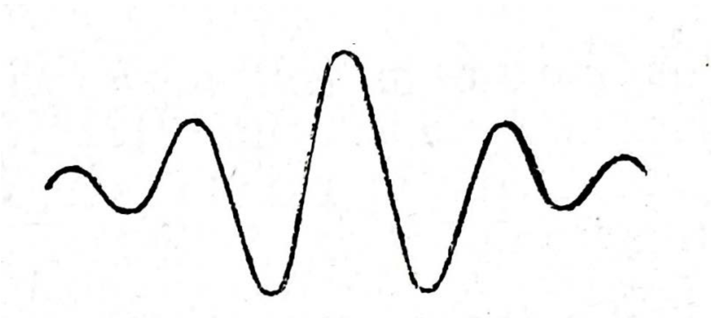
$$\therefore\ v_{gr} = \frac{\frac{1}{2}d\omega}{\frac{1}{2}dk} = \frac{d\omega}{dk}$$

Since $\omega = ck$, we have

$$\frac{d\omega}{dk} = c + k\frac{dc}{dk}$$

Therefore, the group velocity, i.e., the velocity with which the maximum of the group moves, is given by

$$v_{gr} = \frac{d\omega}{dk} = c + k\frac{dc}{dk}$$



Now, $k = \frac{2\pi}{\lambda}$, or $\lambda = \frac{2\pi}{k}$.

$$\therefore\ \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

$$\therefore\ \begin{aligned} v_{gr} &= c + k\frac{dc}{dk} = c + k\frac{dc}{d\lambda}\frac{d\lambda}{dk} = c + k\frac{dc}{d\lambda}\left(-\frac{2\pi}{k^2}\right) \\ &= c - \frac{2\pi}{k}\frac{dc}{d\lambda} = c - \lambda\frac{dc}{d\lambda} \end{aligned}$$

$$\therefore\ v_{gr} = c - \lambda \frac{dc}{d\lambda}$$

In the dispersive medium, phase velocity c changes with the wavelength λ. In a non-dispersive medium, phase velocity c is the same for all wavelengths, i.e. $\frac{dc}{d\lambda} = 0$. Therefore, in a non-dispersive medium, we have $v_{gr} = c$, i.e. group velocity and phase velocity are equal.