Vibrations

Periodic motion:

When a body describes the same path repeatedly in some definite interval of time, its motion is called periodic and the time taken by the body to complete its path once is called its period.

Example:

(i) Motion of the earth round the sun is a periodic motion, and its period is one year.

(ii) The rotation of the earth about its own axis is a periodic motion, period being 24 hours.

(iii) The motion of minute hand of a clock is periodic motion, period being 1 hour.

(iv) Motion of a clock-pendulum is a periodic motion.

Vibratory or oscillatory motion:

The periodic motion of a body is said to be vibratory or oscillatory when it is reversed in direction after a definite interval of time.

Example:

The to- and fro motion of a pendulum is an example of oscillatory motion.

Note: All oscillatory motion is periodic but all periodic motions are not oscillatory.

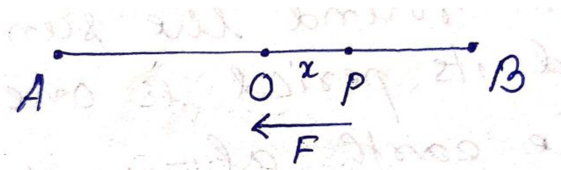
Simple harmonic motion:

It is the simplest kind of oscillatory motion of a body. It has the following characteristics:

(i) The motion is oscillatory and it repeats after equal interval of time

(ii) The restoring force acting on the body (or acceleration of the body) is always proportional to its displacement (measured along its path) from some fixed point on its path called its mean position.

(iii) The restoring force on the body or its acceleration is always directed towards its mean position.



Let a body oscillates between two points A and B along the straight line AB. O be its mean position. At any instant t let the position of the body is at P so that the displacement is $OP = x$. If the force acting on the body at that instant is F then according to the definition of SHM

$F \propto -x$

$\therefore\ F = mf = -sx$

where s is a constant and is called the spring constant or the stiffness constant. The negative sign indicates that the acceleration is in the opposite direction to displacement. The force acting on a body executing SHM acts towards the mean position. That is why it is called the restoring force.

Example: Simple harmonic oscillator when the amplitude of oscillation is small.

Definition of few terms of SHM:

(i) Complete oscillation:

When a particle in SHM starting from any position in its path comes back to the same path after completing the path, we say that it has completed an oscillation.

(ii) Amplitude of oscillation (a):

The maximum displacement from its mean position on both sides of the mean position is called the amplitude of oscillation.

Unit- unit of displacement

(iii) Time period (T):

Time taken by the particle in SHM to complete one complete oscillation is called time period of oscillation.

Unit- unit of time (sec)

(iv) Frequency (n):

Number of complete oscillations per second executed by an oscillatory body is called frequency of oscillation.

Unit- Hertz (Hz)

(v) Phase:

The phase of a vibrating body at any instant determines the state of displacement and motion of the particle at that instant.

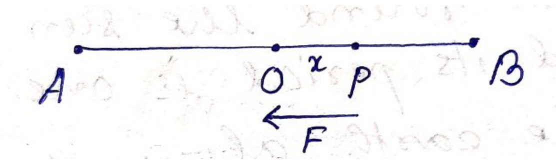
The phase of the vibrating body specifies both the displacement and its direction of motion at any instant of time.

(vi) Epoch:

The initial phase of vibrating body is called epoch.

The phase changes with time but the epoch remains constant with time.

Differential equation of Simple harmonic motion (SHM) and its solution:



Let a particle of mass m at any instant of time t has displacement x from the mean position. If the restoring force acting on the particle is proportional to the displacement and in a direction opposite to it, we can write the equation of motion as

$$F = m\frac{d^2x}{dt^2} = -sx$$

where s is a constant, called the stiffness constant or spring constant which is the force required to produce unit displacement of the particle from its mean position.

$$\therefore\ m\frac{d^2x}{dt^2} = -sx$$

Or, $\dfrac{d^2x}{dt^2} = -\dfrac{s}{m}x$

$$\therefore\ \frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ ....(1)}$$

where $\omega = \sqrt{\dfrac{s}{m}}$, called the angular frequency of vibration.

This is the differential equation of SHM. It is a second order differential equation.

Let $x(t) = Ae^{\alpha t}$, where A, α are constants, be the solution of eq. (1).

$\dfrac{dx}{dt} = A\alpha e^{\alpha t}$ and $\dfrac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$

$$\therefore A\alpha^2 e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

Or, $(\alpha^2 + \omega^2)Ae^{\alpha t} = 0$

$\therefore \alpha^2 + \omega^2 = 0$ [$\because Ae^{\alpha t} \neq 0$ for any value of t]

$\therefore \alpha = \pm i\omega$, where $i = \sqrt{-1}$.

Hence the solution of eq. (1) may be $x = A_1 e^{i\omega t}$ or $x = A_2 e^{-i\omega t}$.

The most general solution of eq. (1) is

$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$ ....(2)

where A and B are two arbitrary constants whose values can be determined from two initials conditions.

$$\therefore \begin{aligned} x &= A(\text{Cos}\,\omega t + i\,\text{Sin}\,\omega t) + B(\text{Cos}\,\omega t - i\,\text{Sin}\,\omega t) \\ &= (A+B)\,\text{Cos}\,\omega t + i(A-B)\,\text{Sin}\,\omega t \end{aligned}$$

A, B may be real or imaginary or they may be complex quantities containing real and imaginary parts.

Since x is real,

$x = a_1\,\text{Cos}\,\omega t + b_1\,\text{Sin}\,\omega t$ ....(3)

where a_1, b_1 are real parts of coefficients of $\text{Cos}\,\omega t$ and $\text{Sin}\,\omega t$.

This is the real solution of the differential equation (1).

Writing $a_1 = a\,\text{Cos}\,\varepsilon$, $b_1 = aSin\varepsilon$ we have

$x(t) = aCos(\omega t - \varepsilon)$ ....(4)

where $a^2 = a_1^{\,2} + b_1^{\,2}$ and $\tan\varepsilon = \dfrac{b_1}{a_1}$.

This is also the solution of eq. (1) in another form.

Hence the maximum possible displacement in the positive direction of X-axis is $x_m = a$ when $\omega t - \varepsilon = 0, 2\pi, 4\pi, ...etc$. The maximum possible displacement in the negative direction of X-axis is $x_m = -a$ when $\omega t - \varepsilon = \pi, 3\pi, 5\pi, ...etc$.

Thus the particle oscillates between two points distance a apart from the mean position. a is called the amplitude of vibration. The same displacement repeats after an interval of time T called the time period given by

$\omega T = 2\pi$

$$\therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

Number of oscillations per second is

$$N = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

Energy of a particle in SHM:

Let the particle of mass m has displacement x at any instant t. The restoring force (opposing displacement) is $-sx$ or $-m\omega^2 x$. If the displacement is increased by dx , work done against this restoring force is $sx.dx$.

Hence the total potential energy is

$$PE = \int_0^x sx.dx = \frac{1}{2}sx^2 = \frac{1}{2}m\omega^2 x^2$$

Kinetic energy at this instant is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\frac{dx}{dt})^2$$

$$v = \frac{dx}{dt} = \frac{d}{dt}(aCos\omega t) = -a\omega Sin\omega t$$

$$v^2 = a^2\omega^2 Sin^2\omega t = a^2\omega^2(1-Cos^2\omega t) = a^2\omega^2(1-\frac{x^2}{a^2}) = \omega^2(a^2-x^2)$$

$$\therefore KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2-x^2)$$

Total energy at any instant t is

$$\mathrm{E} = \mathrm{KE} + \mathrm{PE} = \frac{1}{2}m\omega^2(a^2-x^2) + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 a^2 = constant$$

Thus, we see that total energy of a particle in SHM is constant.