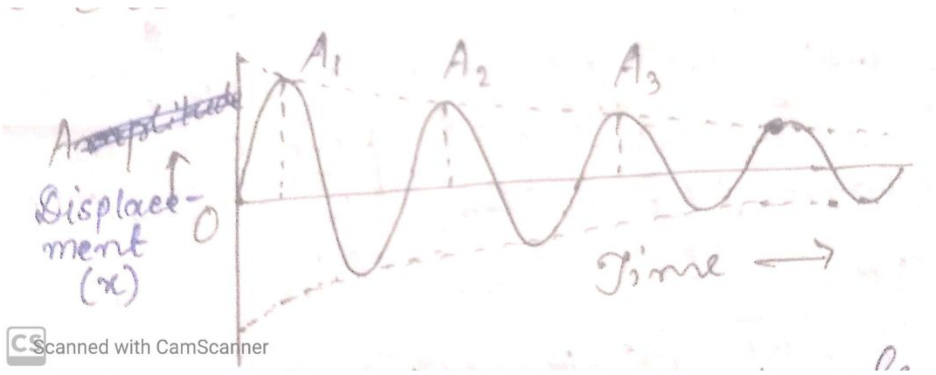
Damped Vibrations

Free Vibration:

After vibrations have been excited and the exciting force is removed, the system continues to vibrate with a frequency characteristics of its own. Such vibrations are called free vibrations. The periodic time of vibration is called the free or natural period. Its reciprocal is called the natural frequency.

However, in all practical cases it is found that when left to itself the vibration of the system gradually diminishes in amplitude and finally dies away. This is because the motion is resisted by various frictional effects, internal as well as external. A vibrating pendulum is damped because resisting forces between its suspension and support and between air and the bob. Vibrations thus hampered from within and without and of gradually diminishing amplitude are called resisted or damped vibrations.



The above figure represents a case of damped simple harmonic motion. At each vibration some energy is lost in overcoming the resisting forces. If the resisting force is large rate of loss of energy is also large. In this case the system quickly comes to rest. With small damping, vibrations continue for a long time, the amplitude diminishes slowly.

Damped vibrations:

Free vibratory motions are opposed by various frictional and viscous forces and finally bring the body to rest. In the simplest case, this resisting or retarding force may be taken to be proportional to the instantaneous velocity of the moving body.

Suppose a particle of mass 'm' is moving along X-axis and subjected to a restoring for proportional to distance from fixed point on the axis and a frictional force proportional to the velocity. Then the equation of motion of such a particle can be written as

$$m\frac{d^2x}{dt^2} = -sx - K\frac{dx}{dt}$$

Where 's' is the stiffness (or spring) constant and 'K' is the resistance coefficient which denotes frictional force per unit velocity.

$$\therefore \frac{d^2x}{dt^2} + \frac{K}{m}\frac{dx}{dt} + \frac{s}{m}x = 0$$

Writing $\frac{s}{m} = \omega^2$ and $\frac{K}{m} = 2b$, we have

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0 \quad \text{............(1)}$$

Let $x(t) = e^{\alpha t}$ be the solution of eq. (1).

$$\therefore \frac{dx}{dt} = \alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

$$\therefore (\alpha^2 + 2b\alpha + \omega^2) e^{\alpha t} = 0$$

$$\therefore \alpha^2 + 2b\alpha + \omega^2 = 0 \; [\because e^{\alpha t} \neq 0]$$

$$\therefore \alpha = \frac{-2b \pm \sqrt{4b^2 - 4.1.\omega^2}}{2.1} = -b \pm \sqrt{b^2 - \omega^2}$$

The general solution of eq. (1) can be written as

$$x(t) = A_1 e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega^2})t} = e^{-bt}(A_1 e^{\sqrt{b^2-\omega^2}\,t} + A_2 e^{-\sqrt{b^2-\omega^2}\,t})$$

where A_1 and A_2 are two arbitrary constants whose values can be determined from the initial conditions.

Case-I <u>Overdamped motion</u>: b>ω (large damping)

Now b>ω

$\therefore b^2 > \omega^2$

$\therefore b^2 > (b^2 - \omega^2)$

$\therefore e^{(-b+\sqrt{b^2-\omega^2})t} = e^{-(b-\sqrt{b^2-\omega^2})t} = e^{-K_1 t}$, where K_1 is a positive quantity.

As $t \to \infty$, $e^{(-b+\sqrt{b^2-\omega^2})t} = e^{-K_1 t} \to 0$

Also $e^{(-b-\sqrt{b^2-\omega^2})t} = e^{-(b+\sqrt{b^2-\omega^2})t} = e^{-K_2 t}$, where K_2 is another positive quantity.

As $t \to \infty$, $e^{(-b-\sqrt{b^2-\omega^2})t} = e^{-K_2 t} \to 0$

Hence $x = A_1 e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega^2})t} \to 0$ as $t \to \infty$. Therefore, x gradually diminishes with time and the motion finally ceases. This motion is called aperiodic or over-damped or dead-beat. It is non-oscillatory and heavily damped.

Particular case:

Let the system have a displacement 'a' and velocity 'v_0' initially.

$\therefore$ x = a and v = v_0 at t = 0.

We have $x = e^{-bt}(A_1 e^{b_1 t} + A_2 e^{-b_1 t})$, where $\sqrt{b^2-\omega^2} = b_1$.

$$\therefore \frac{dx}{dt} = -be^{-bt}(A_1 e^{b_1 t} + A_2 e^{-b_1 t}) + e^{-bt}(A_1 b_1 e^{b_1 t} - A_2 b_1 e^{-b_1 t})$$

At t = 0, x = a and v = $\frac{dx}{dt}$ = v_0

$\therefore A_1 + A_2 = a$ and $-b(A_1 + A_2) + b_1(A_1 - A_2) = v_0$

Or, $-ba + b_1(A_1 - A_2) = v_0$

$$\therefore A_1 - A_2 = \frac{v_0 + ba}{b_1}$$

Solving for A_1 and A_2 we get

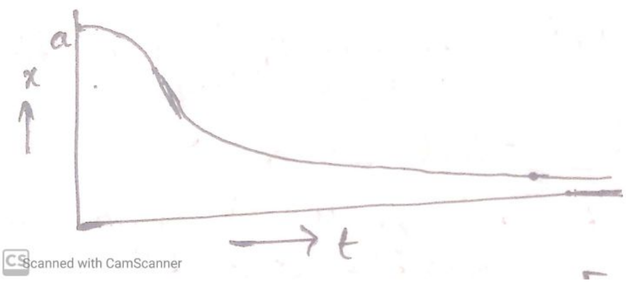
$$\therefore A_1 = \frac{a + \frac{v_0 + ba}{b_1}}{2} \text{ and } A_2 = \frac{a - \frac{v_0 + ba}{b_1}}{2}$$

$$\therefore A_1 = \frac{a}{2}\left(1 + \frac{b + v_0/a}{b_1}\right) \text{ and } A_2 = \frac{a}{2}\left(1 - \frac{b + v_0/a}{b_1}\right)$$

$$\therefore x = \frac{a}{2}e^{-bt}\left[\left(1 + \frac{b + v_0/a}{\sqrt{b^2-\omega^2}}\right)e^{\sqrt{b^2-\omega^2}\,t} + \left(1 - \frac{b + v_0/a}{\sqrt{b^2-\omega^2}}\right)e^{-\sqrt{b^2-\omega^2}\,t}\right]$$

If the particle is displaced to 'a' and then released so that $\frac{dx}{dt}$ = v_0 = 0 at t = 0.

$$\therefore x = \frac{a}{2}e^{-bt}\left[\left(1 + \frac{b}{\sqrt{b^2-\omega^2}}\right)e^{\sqrt{b^2-\omega^2}\,t} + \left(1 - \frac{b}{\sqrt{b^2-\omega^2}}\right)e^{-\sqrt{b^2-\omega^2}\,t}\right]$$



Case-II Under-damped Oscillation: b<ω (small damping)

Now b<ω

$\therefore b^2<\omega^2$

$$\therefore \sqrt{b^2-\omega^2}=i\sqrt{\omega^2-b^2}$$

$$\therefore x=e^{-\mathrm{bt}}(A_1e^{i\sqrt{\omega^2-b^2}\,\mathrm{t}}+A_2e^{-i\sqrt{\omega^2-b^2}\,\mathrm{t}})$$

$$\therefore \mathrm{x}=e^{-\mathrm{bt}}[(A_1+A_2)\mathrm{Cos}\sqrt{\omega^2-b^2}\;\mathrm{t}+i(A_1-A_2)\mathrm{Sin}\sqrt{\omega^2-b^2}\;\mathrm{t}]$$

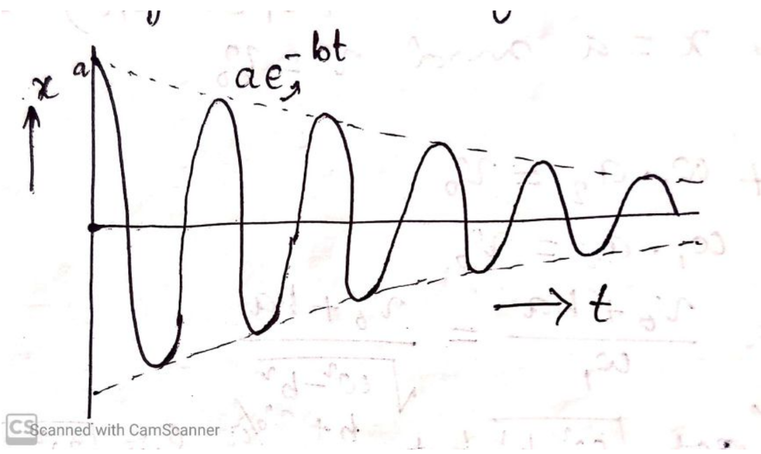
Now A_1 and A_2 may both contain real as well as imaginary parts. Let a_1 and a_2 be the real parts of coefficient s of $\mathrm{Cos}\sqrt{\omega^2-b^2}\;\mathrm{t}$ and $\mathrm{Sin}\sqrt{\omega^2-b^2}\;\mathrm{t}$ respectively.

$\therefore \mathrm{x}=e^{-\mathrm{bt}}(a_1\,\mathrm{Cos}\sqrt{\omega^2-b^2}\;\mathrm{t}+a_2\,\mathrm{Sin}\sqrt{\omega^2-b^2}\;\mathrm{t})=e^{-\mathrm{bt}}(a_1\,\mathrm{Cos}\,\omega_1\,\mathrm{t}+a_2\,\mathrm{Sin}\,\omega_1\,\mathrm{t})$, where $\sqrt{\omega^2-b^2}=\omega_1$.

This is the equation of damped oscillatory motion. Constants a_1 and a_2 can be found from the initial conditions.

If we put $a_1=\mathrm{R}\,Cos\theta$ and $a_2=\mathrm{R}\,Sin\theta$, we can write

$$x=\mathrm{R}\,e^{-bt}Cos(\sqrt{w^2-b^2}\;\mathrm{t}-\theta)$$



Particular case:

Let the initial displacement and velocity be ‘a’ and ‘v_0’ respectively.

Now $\mathrm{x}=e^{-\mathrm{bt}}(a_1\,\mathrm{Cos}\,\omega_1\,\mathrm{t}+a_2\,\mathrm{Sin}\,\omega_1\,\mathrm{t})$

$$\therefore \frac{dx}{dt}=-be^{-\mathrm{bt}}(a_1\,\mathrm{Cos}\,\omega_1\,\mathrm{t}+a_2\,\mathrm{Sin}\,\omega_1\,\mathrm{t})+e^{-\mathrm{bt}}(-a_1\omega_1\,\mathrm{Sin}\,\omega_1\,\mathrm{t}+a_2\omega_1\,\mathrm{Cos}\,\omega_1\,\mathrm{t})$$

Now at t = 0, x = a and v = v_0

$\therefore$ $a = a_1$ and $-b a_1 + a_2\omega_1 = v_0$

Or, $-ba + a_2\omega_1 = v_0$

$$\therefore a_2 = \frac{v_0 + ba}{\omega_1} = \frac{v_0 + ba}{\sqrt{\omega^2 - b^2}}$$

$$\therefore x = ae^{-bt}(\text{Cos}\sqrt{\omega^2 - b^2}\ \text{t} + \frac{b + v_0/a}{\sqrt{\omega^2 - b^2}}\text{Sin}\sqrt{\omega^2 - b^2}\ \text{t})$$

Let $a = \text{RCos}\,\theta$ and $\frac{v_0 + ba}{\sqrt{\omega^2 - b^2}} = \text{RSin}\,\theta$.

$$\therefore x = \text{R}\,e^{-bt}(\text{Cos}\sqrt{\omega^2 - b^2}\ \text{t} - \theta)$$

Now, $\text{R}^2\text{Cos}^2\theta + \text{R}^2\,\text{Sin}^2\theta = \text{R}^2 = a^2 + \frac{(v_0 + ba)^2}{\omega^2 - b^2}$

$\therefore \text{R} = \sqrt{\frac{a^2\omega^2 + v_0{}^2 + 2bav_0}{w^2 - b^2}}$ and $\tan\theta = \frac{b + v_0/a}{\sqrt{w^2 - b^2}}$.

If the body be displaced to ‘a’ and then released. Then at t = 0, x = a and $v = \frac{dx}{dt} = 0$.

$\therefore v_0 = 0$

$\therefore \text{R}' = \frac{a\omega}{\sqrt{w^2 - b^2}}$ and $\tan\theta' = \frac{b}{\sqrt{w^2 - b^2}}$

$$\therefore x = \frac{a\omega}{\sqrt{w^2 - b^2}}e^{-bt}Cos(\sqrt{w^2 - b^2}\ \text{t} - \theta')$$

Thus the displacement x as given in the above equations express damped harmonic motion, the amplitude decreasing exponentially with time. The time period $\text{T} = \frac{2\pi}{\sqrt{w^2 - b^2}}$ is slightly greater than the time period for free or natural vibration which is $\text{T} = \frac{2\pi}{\omega}$.

Now, $\omega^2 = \frac{s}{m}$ and $b^2 = \frac{K^2}{4m^2}$. b^2 is a term of smallness of second order in comparison to ω^2 and can in most cases be neglected. Thus the time period of oscillation is very slightly affected by damping of ordinary magnitude.

Let x_0, x_1, x_2, x_3,.........etc. be the maximum displacements of the system in both directions at time given by ω_1t-θ = $\sqrt{w^2-b^2}\,t-\theta$ = 0, π, 2π, 3π,..... etc. respectively.

$$\therefore x_0 = \mathrm{R}\,e^{-\frac{b\theta}{\omega_1}},\ x_1 = \mathrm{R}\,e^{-\frac{b(\theta+\pi)}{\omega_1}},\ x_2 = \mathrm{R}\,e^{-\frac{b(\theta+2\pi)}{\omega_1}},\ x_3 = \mathrm{R}\,e^{-\frac{b(\theta+3\pi)}{\omega_1}} \text{ etc.}$$

Thus neglecting the sign of displacements,

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = ... = e^{\frac{b\pi}{\omega_1}} = e^{\frac{b}{2}\cdot\frac{2\pi}{\omega_1}} = e^{\frac{bT}{2}}\ [\because \mathrm{T} = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{w^2-b^2}}]$$

Hence $b = \frac{2}{\mathrm{T}}\log\frac{x_0}{x_1} = \frac{2}{\mathrm{T}}\log\frac{x_1}{x_2} = \frac{2}{\mathrm{T}}\log\frac{x_2}{x_3} = \frac{2}{\mathrm{T}}\log\frac{x_3}{x_4}.........etc.$

Thus the damping coefficient can be found out from an experimental measurement of consecutive amplitudes.

Case-III Critically damped motion: b → ω

We have,

$x(t) = A_1 e^{(-\mathrm{b}+\sqrt{b^2-\omega^2})\mathrm{t}} + A_2 e^{(-\mathrm{b}-\sqrt{b^2-\omega^2})\mathrm{t}} = e^{-\mathrm{bt}}(A_1 e^{\beta\mathrm{t}} + A_2 e^{-\beta\mathrm{t}})$, where $\sqrt{b^2-\omega^2} = \beta$ is a very small quantity.

$$x = e^{-\mathrm{bt}}(A_1 e^{\beta\mathrm{t}} + e^{-\beta\mathrm{t}}) = e^{-\mathrm{bt}}\{A_1(1+\beta t) + A_2(1-\beta t)\} = e^{-\mathrm{bt}}\{(A_1+A_2) + (A_1-A_2)\beta t\}$$
$$= e^{-\mathrm{bt}}(A+Bt)$$

where $A_1 + A_2 = A$ and $(A_1 - A_2)\beta = B$.

The above equation of x is that of critically damped motion.

Particular case:

Suppose the particle has displacement x = a and velocity v = v_0 at t = 0.Then we have,

$$\mathrm{x} = x = \frac{a}{2}e^{-\mathrm{bt}}[(1+\frac{b+\mathrm{v}_0/a}{\sqrt{b^2-\omega^2}})e^{\sqrt{b^2-\omega^2}\,\mathrm{t}} + (1-\frac{b+\mathrm{v}_0/a}{\sqrt{b^2-\omega^2}})e^{-\sqrt{b^2-\omega^2}\,\mathrm{t}}]$$
$$= \frac{a}{2}e^{-\mathrm{bt}}[(1+\frac{b+\mathrm{v}_0/a}{\sqrt{b^2-\omega^2}})(1+\sqrt{b^2-\omega^2}\ \mathrm{t}) + (1-\frac{b+\mathrm{v}_0/a}{\sqrt{b^2-\omega^2}})(1-\sqrt{b^2-\omega^2}\ \mathrm{t})$$

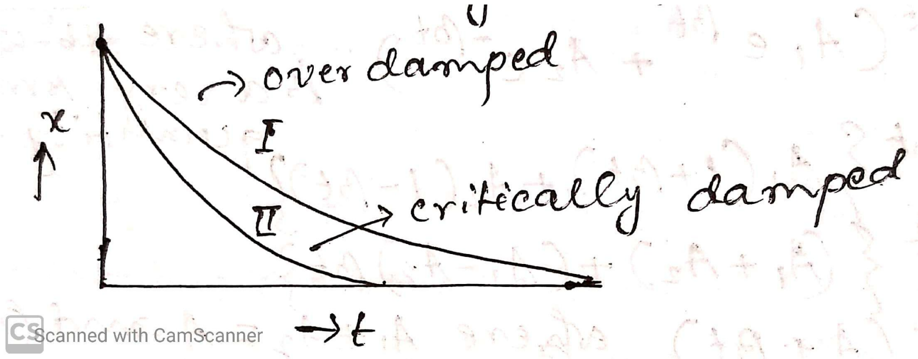
[Expanding and neglecting higher order terms as b → ω]

$$x = \frac{a}{2} e^{-\mathrm{bt}} [2 + \frac{b + \mathrm{v}_0 / a}{\sqrt{b^2 - \omega^2}} .2\sqrt{b^2 - \omega^2} \ \mathrm{t}]$$
$$= ae^{-\mathrm{bt}} [1 + (b + \mathrm{v}_0 / a)\,\mathrm{t}]$$
$$= e^{-\mathrm{bt}} [a + (\mathrm{v}_0 + ba)\,\mathrm{t}]$$
$$= e^{-\mathrm{bt}} (A + Bt)$$

where $A = a$ and $\mathrm{v}_0 + ba = B$.

$$\therefore\ x = e^{-\mathrm{bt}} (A + Bt)$$

The motion represented by this equation is also aperiodic, but x reaches the final value more quickly than when b > ω. This case is known in mechanics as critically damped motion.



When the particle under critical damping is given an initial velocity v_0 while x = 0 i.e. x = 0, $\mathrm{v} = \mathrm{v}_0$ at t = 0.

$\therefore$ A = 0 and $B = \mathrm{v}_0$

$$\therefore\ x = \mathrm{v}_0\, te^{-\mathrm{bt}}$$

$$\therefore\ \frac{dx}{dt} = \mathrm{v}_0\, e^{-\mathrm{bt}} - \mathrm{v}_0\, te^{-\mathrm{bt}} b = \mathrm{v}_0\, e^{-\mathrm{bt}} (1 - bt)$$

Thus the body comes to rest for the first time when $\frac{dx}{dt} = \mathrm{v} = 0$.

$$\therefore\ t = \frac{1}{b}$$

This time is independent of the initial velocity. The maximum displacement it undergoes is $x_{\max} = \frac{\mathrm{v}_0}{eb}$.

Time constant or modulus of decay:

This is the time in which the amplitude decreases to $\frac{1}{e}$ of its initial value. We have,

$$a(t) = ae^{-bt}$$

Let in time τ amplitude falls $\frac{1}{e}$ times its initial value which is 'a'.

$\therefore a(0) = a$ and $a(\tau) = ae^{-b\tau}$.

According to the definition of time constant,

$$\frac{a(\tau)}{a(0)} = \frac{1}{e}$$

$$\therefore \frac{ae^{-b\tau}}{a} = \frac{1}{e}$$

Or, $e^{-b\tau} = e^{-1}$

Or, $b\tau = 1$

$$\therefore \tau = \frac{1}{b} = \frac{2m}{K} \quad [\because \frac{K}{m} = 2b]$$

The heavier the mass or smaller the damping, the larger is the time constant (τ). τ gives the measure of how quickly the motion is damped by the resisting force.

Energy in forced vibrations:

Let x be the displacement of the particle at any instant t. The restoring force on the particle is sx (s is the spring/stiffness constant). The potential energy of the particle at any instant t is

$$PE = \frac{1}{2}sx^2$$

The kinetic energy at the same instant t is

$$KE = \frac{1}{2}m(\frac{dx}{dt})^2$$

If the particle further describes an element of displacement dx, the loss of total energy (KE+PE) of the particle will be equal to the work done against the frictional force.

The frictional force on the particle is $K\frac{dx}{dt}$. Hence the work done against the frictional force is

$K\frac{dx}{dt}dx$.

$$\therefore -d\{\frac{1}{2}sx^2+\frac{1}{2}m(\frac{dx}{dt})^2\}=K\frac{dx}{dt}dx$$

The rate loss of energy i.e. loss of energy per unit time is

$$-\frac{d}{dt}\{\frac{1}{2}sx^2+\frac{1}{2}m(\frac{dx}{dt})^2\}=K\frac{dx}{dt}\frac{dx}{dt}$$

Or, $\frac{1}{2}s.2x.\frac{dx}{dt}+\frac{1}{2}m.2\frac{dx}{dt}.\frac{d^2x}{dt^2}=-K(\frac{dx}{dt})^2$

Or, $\frac{dx}{dt}\{\mathrm{m}\frac{d^2x}{dt^2}+K\frac{dx}{dt}+sx\}=0$

$\therefore \mathrm{m}\frac{d^2x}{dt^2}+K\frac{dx}{dt}+sx=0$ $[\because \frac{dx}{dt}\neq 0$ for all values of t]

This is the required equation of damped motion.

Average kinetic energy and potential energy:

For small damping (underdamped motion) the displacement of the particle at any instant is given by

$$x=ae^{-\mathrm{bt}}\,\mathrm{Cos}(\sqrt{{\omega_0}^2-b^2}\,\mathrm{t}-\theta)=ae^{-\mathrm{bt}}\,\mathrm{Cos}(\omega\,\mathrm{t}-\theta),$$

where a is the initial displacement of the particle, θ is the initial phase, ω_0 the frequency of natural/free vibration and $\omega=\sqrt{{\omega_0}^2-b^2}$ is the frequency of damped vibration.

The kinetic energy of the particle is given by

$$KE=\frac{1}{2}m(\frac{dx}{dt})^2=\frac{1}{2}ma^2[-\mathrm{b}\,e^{-\mathrm{bt}}\,\mathrm{Cos}(\omega\,\mathrm{t}-\theta)-e^{-\mathrm{bt}}\omega\,\mathrm{Sin}(\omega\,\mathrm{t}-\theta)]^2$$
$$=\frac{1}{2}ma^2e^{-2\mathrm{bt}}[\mathrm{bCos}(\omega\,\mathrm{t}-\theta)+\omega\,\mathrm{Sin}(\omega\,\mathrm{t}-\theta)]^2$$
$$=\frac{1}{2}ma^2e^{-2\mathrm{bt}}[\mathrm{b}^2\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta)+\omega^2\,\mathrm{Sin}^2(\omega\,\mathrm{t}-\theta)+2\,\mathrm{b}\,\omega\,\mathrm{Sin}(\omega\,\mathrm{t}-\theta)\mathrm{Cos}(\omega\,\mathrm{t}-\theta)]$$
$$=\frac{1}{2}ma^2e^{-2\mathrm{bt}}[\mathrm{b}^2\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta)+\omega^2\,\mathrm{Sin}^2(\omega\,\mathrm{t}-\theta)+\mathrm{b}\,\omega\,\mathrm{Sin}\,2(\omega\,\mathrm{t}-\theta)]$$

The potential energy of the particle is

$$PE = \frac{1}{2}sx^2 = \frac{1}{2}sa^2e^{-2\mathrm{bt}}\,\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta) = \frac{1}{2}m\omega_0{}^2a^2e^{-2\mathrm{bt}}\,\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta)$$

Hence the total energy is given by

$$\mathrm{E} = \mathrm{KE}+\mathrm{PE}$$
$$= \frac{1}{2}ma^2e^{-2\mathrm{bt}}[\mathrm{b}^2\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta)+\omega^2\,\mathrm{Sin}^2(\omega\,\mathrm{t}-\theta)+\mathrm{b}\,\omega\,\mathrm{Sin}\,2(\omega\,\mathrm{t}-\theta)+\omega_0{}^2\,\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta)]$$

Hence the total energy of the particle in damped vibration is not constant, it changes with time.

Average kinetic energy is

$$< KE >= \frac{\int_0^T \frac{1}{2}ma^2e^{-2\mathrm{bt}}[\mathrm{b}^2\mathrm{Cos}^2(\omega\,\mathrm{t}-\theta)+\omega^2\,\mathrm{Sin}^2(\omega\,\mathrm{t}-\theta)+\mathrm{b}\,\omega\,\mathrm{Sin}\,2(\omega\,\mathrm{t}-\theta)]dt}{\int_0^T dt}$$
$$= \frac{1}{T}.\frac{1}{2}ma^2e^{-2\mathrm{bt}}[\mathrm{b}^2.\frac{T}{2}+\omega^2.\frac{T}{2}]$$

[$e^{-2\mathrm{bt}}$ is taken to be constant since the amplitude of oscillation $ae^{-\mathrm{bt}}$ having term $e^{-2\mathrm{bt}}$ does not change in the cycle of motion]

$$< KE >= \frac{1}{4}ma^2e^{-2\mathrm{bt}}(\mathrm{b}^2+\omega^2)$$
$$= \frac{1}{4}ma^2(\mathrm{b}^2+\omega_0{}^2-b^2)e^{-2\mathrm{bt}}$$
$$= \frac{1}{4}m\omega_0{}^2a^2e^{-2\mathrm{bt}}$$

The average potential energy is

$$< PE >= \frac{\int_0^T \frac{1}{2}m\omega_0{}^2a^2e^{-2\mathrm{bt}}Cos^2(\omega\,\mathrm{t}-\theta)dt}{\int_0^T dt}$$
$$= \frac{1}{T}.\frac{1}{2}m\omega_0{}^2a^2e^{-2\mathrm{bt}}.\frac{T}{2}$$
$$= \frac{1}{4}m\omega_0{}^2a^2e^{-2\mathrm{bt}}$$

Hence the average kinetic energy and average potential energy over a complete cycle is equal.

The average total energy is given by

$$< E >= \frac{1}{4} m\omega_0{}^2 a^2 e^{-2\mathrm{bt}} + \frac{1}{4} m\omega_0{}^2 a^2 e^{-2\mathrm{bt}}$$
$$= \frac{1}{2} m\omega_0{}^2 a^2 e^{-2\mathrm{bt}}$$
$$= \frac{1}{2} m\omega_0{}^2 a^2 e^{-2.\frac{K}{2m}.\mathrm{t}}$$
$$= \frac{1}{2} m\omega_0{}^2 a^2 e^{-\frac{K}{m}.\mathrm{t}}$$

The average loss of energy against the resisting force over a complete cycle is

$$< p(t) >= \frac{\int_0^T K(\frac{dx}{dt})^2 dt}{\int_0^T dt}$$
$$= \frac{1}{T}.\int_0^T K(\frac{dx}{dt})^2 dt$$
$$= \frac{2K}{m}.\frac{1}{T}.\int_0^T \frac{1}{2} m(\frac{dx}{dt})^2 dt$$
$$= \frac{2K}{m}.< KE >$$
$$= \frac{2K}{m}.\frac{1}{4} m\omega_0{}^2 a^2 e^{-2\mathrm{bt}}$$
$$= \frac{K}{m}.\frac{1}{2} m\omega_0{}^2 a^2 e^{-2\mathrm{bt}}$$
$$= \frac{K}{m}.< E >$$

The quality factor Q is given by

$$Q = 2\pi \left(\frac{Energy\ stored}{Time\ period\ X\ Energy\ loss\ in\ unit\ time} \right)$$
$$= 2\pi \frac{< E >}{T.< p(t) >}$$
$$= 2\pi.\frac{< E >}{T.\frac{K}{m}.< E >}$$
$$= \frac{2\pi}{T}.\frac{m}{K}$$
$$= \frac{\omega m}{K}$$

We can see that for small damping is (low K) i.e. weak damping the quality factor is high.