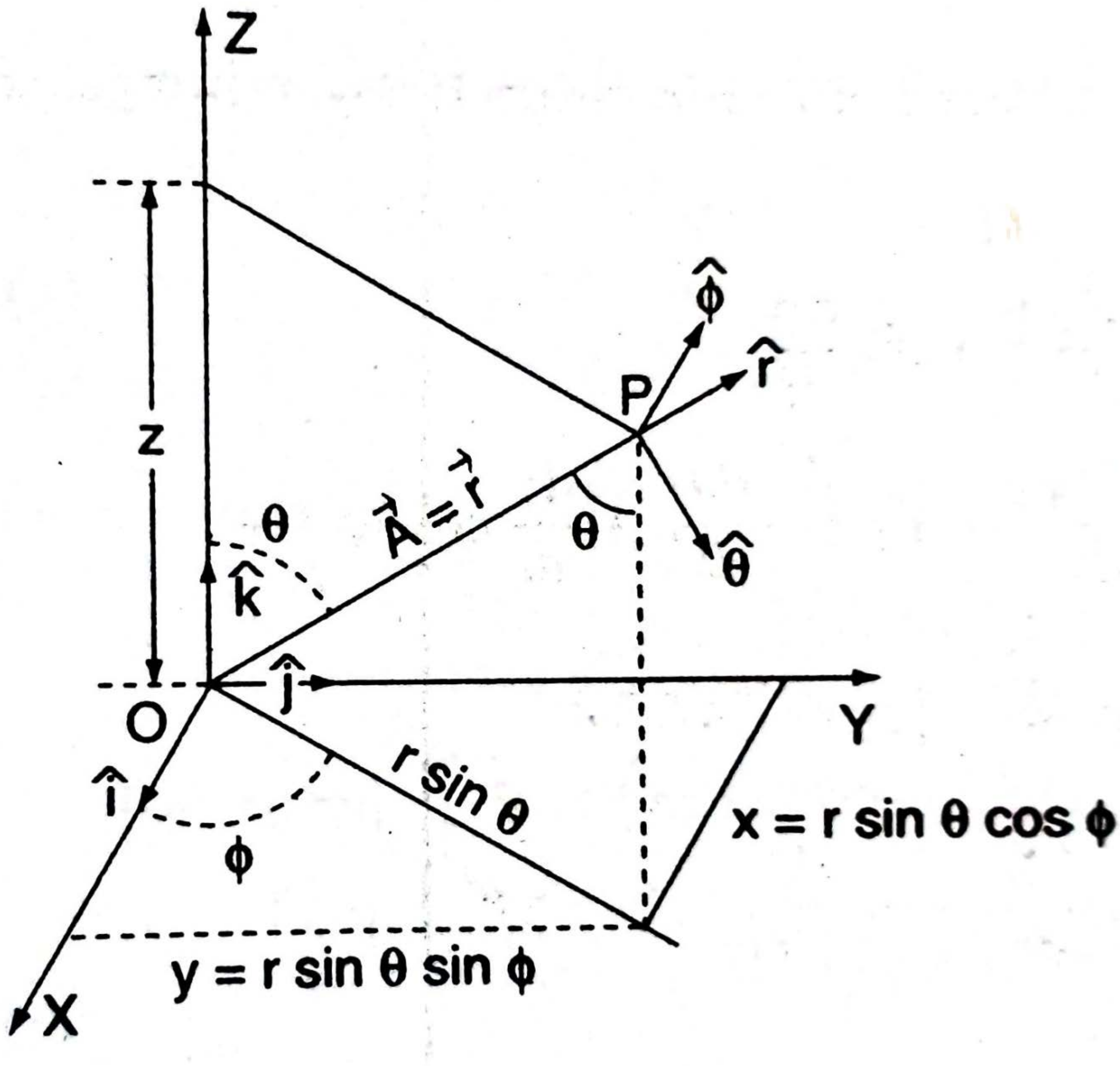
Velocity and acceleration of a particle in spherical polar coordinates:



In spherical polar coordinate system, coordinates of a point P is represented by (r,θ,ϕ) and they are related to the Cartesian coordinates (x,y,z) by the following relations

$x = rSin\theta Cos\phi, \mathrm{y} = rSin\theta Sin\phi, \mathrm{z} = rCos\theta$ ....(1)

The position vector of a particle which is at point P at iny instant is given by,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = rSin\theta Cos\phi\hat{i} + rSin\theta Sin\phi\,\hat{j} + rCos\theta\hat{k}$ ....(2)

$\frac{\partial\vec{r}}{\partial r}$ is a vector in the direction of increasing r and a unit vector in this direction is given by,

$\hat{r} = \frac{\partial\vec{r}}{\partial r} / \left|\frac{\partial\vec{r}}{\partial r}\right|$ ....(3)

Now, $\frac{\partial \vec{r}}{\partial r} = Sin\theta Cos\phi\hat{i} + Sin\theta Sin\phi\hat{j} + Cos\theta\hat{k}$ and

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{Sin^2\theta Cos^2\phi + Sin^2\theta Sin^2\phi + Cos^2\theta}$$
$$= \sqrt{Sin^2\theta(Cos^2\phi + Sin^2\phi) + Cos^2\theta}$$
$$= \sqrt{Sin^2\theta + Cos^2\theta} = 1$$

$\therefore \hat{r} = Sin\theta Cos\phi\hat{i} + Sin\theta Sin\phi\hat{j} + Cos\theta\hat{k}$ ....(4)

Similarly, $\hat{\theta} = \frac{\partial \vec{r}}{\partial \theta} / \left|\frac{\partial \vec{r}}{\partial \theta}\right|$ is a unit vector in the direction of increasing θ and $\hat{\phi} = \frac{\partial \vec{r}}{\partial \phi} / \left|\frac{\partial \vec{r}}{\partial \phi}\right|$ is a unit vector in the direction of increasing ϕ.

Now, $\frac{\partial \vec{r}}{\partial \theta} = rCos\theta Cos\phi\hat{i} + rCos\theta Sin\phi\hat{j} - rSin\theta\hat{k}$ and

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2Cos^2\theta Cos^2\phi + r^2Cos^2\theta Sin^2\phi + r^2Sin^2\theta}$$
$$= \sqrt{r^2Cos^2\theta(Cos^2\phi + Sin^2\phi) + r^2Sin^2\theta}$$
$$= \sqrt{r^2(Sin^2\theta + Cos^2\theta)} = r$$

$\therefore \hat{\theta} = Cos\theta Cos\phi\hat{i} + Cos\theta Sin\phi\hat{j} - Sin\theta\hat{k}$ ....(5)

$\frac{\partial \vec{r}}{\partial \phi} = -rSin\theta Sin\phi\hat{i} + rSin\theta Cos\phi\hat{j}$ and

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{r^2Sin^2\theta Sin^2\phi + r^2Sin^2\theta Cos^2\phi}$$
$$= \sqrt{r^2Sin^2\theta(Sin^2\phi + Cos^2\phi)}$$
$$= rSin\theta$$

$\therefore \hat{\phi} = -Sin\phi\hat{i} + Cos\phi\hat{j}$ ....(6)

Now,

$$\frac{d\hat{r}}{dt} = Cos\theta\frac{d\theta}{dt}Cos\phi\hat{i} - Sin\theta Sin\phi\frac{d\phi}{dt}\hat{i} + Cos\theta\frac{d\theta}{dt}Sin\phi\hat{j} + Sin\theta Cos\phi\frac{d\phi}{dt}\hat{j} - Sin\theta\frac{d\theta}{dt}\hat{k}$$
$$= \frac{d\theta}{dt}\left(Cos\theta Cos\phi\hat{i} + Cos\theta Sin\phi\hat{j} - Sin\theta\hat{k}\right) + \frac{d\phi}{dt}\left(-Sin\theta Sin\phi\hat{i} + Sin\theta Cos\phi\hat{j}\right)$$
$$= \frac{d\theta}{dt}\hat{\theta} + Sin\theta\frac{d\phi}{dt}\hat{\phi}........................(7)$$

$$\frac{d\hat{\theta}}{dt} = -Sin\theta\frac{d\theta}{dt}Cos\phi\hat{i} - Cos\theta Sin\phi\frac{d\phi}{dt}\hat{i} - Sin\theta\frac{d\theta}{dt}Sin\phi\hat{j} + Cos\theta Cos\phi\frac{d\phi}{dt}\hat{j} - Cos\theta\frac{d\theta}{dt}\hat{k}$$

$$= \frac{d\theta}{dt}\left(-Sin\theta Cos\phi\hat{i} - Sin\theta Sin\phi\hat{j} - Cos\theta\hat{k}\right) + \frac{d\phi}{dt}\left(-Cos\theta Sin\phi\hat{i} + Cos\theta Cos\phi\hat{j}\right)$$

$$= -\frac{d\theta}{dt}\hat{r} + Cos\theta\frac{d\phi}{dt}\hat{\phi} \text{.......................(8)}$$

$$\frac{d\hat{\phi}}{dt} = -Cos\phi\frac{d\phi}{dt}\hat{i} - Sin\phi\frac{d\phi}{dt}\hat{j}$$

$$= -\frac{d\phi}{dt}\left(Cos\phi\hat{i} + Sin\phi\hat{j}\right) \text{..................(9)}$$

Again,

$$Sin\theta\hat{r} + Cos\theta\hat{\theta} = Sin\theta\left(Sin\theta Cos\phi\hat{i} + Sin\theta Sin\phi\hat{j} + Cos\theta\hat{k}\right)$$

$$+Cos\theta\left(Cos\theta Cos\phi\hat{i} + Cos\theta Sin\phi\hat{j} - Sin\theta\hat{k}\right)$$

$$= \left(Sin^2\theta + Cos^2\theta\right)Cos\phi\hat{i} + \left(Sin^2\theta + Cos^2\theta\right)Sin\phi\hat{j}$$

$$= Cos\phi\hat{i} + Sin\phi\hat{j}$$

$$\therefore \frac{d\hat{\phi}}{dt} = -\frac{d\phi}{dt}\left(Sin\theta\hat{r} + Cos\theta\hat{\theta}\right) \text{....................................(10)}$$

Now, $\vec{r} = r\hat{r}$ ....(11)

Velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r\hat{r}\right) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \frac{dr}{dt}\hat{r} + r\left(\frac{d\theta}{dt}\hat{\theta} + Sin\theta\frac{d\phi}{dt}\hat{\phi}\right)$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + rSin\theta\frac{d\phi}{dt}\hat{\phi}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + rSin\theta\dot{\phi}\hat{\phi} \text{..........................(11)}$$

Acceleration:

$$\hat{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + rSin\theta\frac{d\phi}{dt}\hat{\phi}\right)$$

$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt} + \frac{dr}{dt}Sin\theta\frac{d\phi}{dt}\hat{\phi} + rCos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}\hat{\phi}$$

$$+ rSin\theta\frac{d^2\phi}{dt^2}\hat{\phi} + rSin\theta\frac{d\phi}{dt}\frac{d\hat{\phi}}{dt}$$

$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\hat{\theta} + Sin\theta\frac{d\phi}{dt}\hat{\phi}\right) + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\left(-\frac{d\theta}{dt}\hat{r} + Cos\theta\frac{d\phi}{dt}\hat{\phi}\right)$$

$$+ \frac{dr}{dt}Sin\theta\frac{d\phi}{dt}\hat{\phi} + rCos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}\hat{\phi} + rSin\theta\frac{d^2\phi}{dt^2}\hat{\phi} + rSin\theta\frac{d\phi}{dt}\left[-\frac{d\phi}{dt}\left(Sin\theta\hat{r} + Cos\theta\hat{\theta}\right)\right]$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - rSin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]\hat{r} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} - rSin\theta Cos\theta\left(\frac{d\phi}{dt}\right)^2\right]\hat{\theta}$$

$$+ \left[rSin\theta\frac{d^2\phi}{dt^2} + 2Sin\theta\frac{dr}{dt}\frac{d\phi}{dt} + 2rCos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}\right]\hat{\phi}$$

$$= \left(\ddot{r} - r\dot{\theta}^2 - rSin^2\theta\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - rSin\theta Cos\theta\dot{\phi}^2\right)\hat{\theta} + \left(rSin\theta\ddot{\phi} + 2Sin\theta\dot{r}\dot{\phi} + 2rCos\theta\dot{\theta}\dot{\phi}\right)\hat{\phi} \quad \text{.....................(12)}$$

Note:

Any arbitrary vector in spherical polar coordinates can be written as,

$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$ ....(13)

Here A_r, A_θ, A_ϕ are the components of $\vec{A}$ in the directions of $\hat{r}, \hat{\theta}, \hat{\phi}$ respectively.

$$\frac{d\vec{A}}{dt} = \frac{dA_r}{dt}\hat{r} + A_r\frac{d\hat{r}}{dt} + \frac{dA_\theta}{dt}\hat{\theta} + A_\theta\frac{d\hat{\theta}}{dt} + \frac{dA_\phi}{dt}\hat{\phi} + A_\phi\frac{d\hat{\phi}}{dt}$$

$$= \frac{dA_r}{dt}\hat{r} + A_r\left(\frac{d\theta}{dt}\hat{\theta} + Sin\theta\frac{d\phi}{dt}\hat{\phi}\right) + \frac{dA_\theta}{dt}\hat{\theta} + A_\theta\left(-\frac{d\theta}{dt}\hat{r} + Cos\theta\frac{d\phi}{dt}\hat{\phi}\right) + \frac{dA_\phi}{dt}\hat{\phi} + A_\phi\left[-\frac{d\phi}{dt}\left(Sin\theta\hat{r} + Cos\theta\hat{\theta}\right)\right]$$

$$= \left(\frac{dA_r}{dt} - A_\theta\frac{d\theta}{dt} - A_\phi Sin\theta\frac{d\phi}{dt}\right)\hat{r} + \left(A_r\frac{d\theta}{dt} + \frac{dA_\theta}{dt} - A_\phi Cos\theta\frac{d\phi}{dt}\right)\hat{\theta} + \left(A_r Sin\theta\frac{d\phi}{dt} + A_\theta Cos\theta\frac{d\phi}{dt} + \frac{dA_\phi}{dt}\right)\hat{\phi}$$

....(14)