Partial Differential Equation

Separation of Variables Technique

Wave Equation in (1 + 1) dimensions (Cartesian co-ordinates) :

$$\partial^2 y/\partial x^2 - 1/c^2\, \partial^2 y/\partial t^2 = 0$$

Assume : y(x, t) is 'separable' in the product form : y(x, t) = f(x) T(t). Subst. in the diff. eqn. :

$$d^2f/dx^2\, T(t) - 1/c^2\, f(x)\, d^2T/dt^2 = 0.$$

Note that f(x) and T(t) are functions of single variables. So their derivatives are **not partial, but**
ordinary derivatives. Divide both sides by y(x, t), i.e., f(x) T(t).

$$\Rightarrow\ f''(x)/f(x) - 1/c^2\, T''(t)/T(t) = 0$$

$$\Rightarrow\ f''(x)/f(x) = 1/c^2\, T''(t)/T(t)$$

Now, a function of 'x' cannot be equal to a function of 't' for all values of x and t (they may, accidentally match at some particular pair of values of x and t), unless both are constant functions. [Note that **'ϕ(x) = constant' is a perfectly valid function.**] So, we conclude :

$f''(x)/f(x) = 1/c^2\, T''(t)/T(t) = C$, where 'C' is called the **'separation constant'**.

If we chose the separation constant to be +ve, we shall have exponential solutions for both f(x) and T(t), but if we chose it to be –ve, we shall get sinusoidal (i.e., periodic) solutions. Suppose, we have the boundary conditions :

(i) y(x, t) = 0 at x = 0 for all values of t,
(ii) y(x, t) = 0 at x = L for all values of t.

This requires the solutions to repeat their values at x = 0 and x = L. So, we choose :

$\mathbf{C = -k^2}$ (i.e., -ve).

$$\Rightarrow f''(x)/f = -k^2,\ \ T''(t)/T = -c^2k^2$$

$$\Rightarrow\ f(x) = A\cos kx + B\sin kx \ \text{ and } \ T(t) = C\cos(ckt) + D\sin(ckt)$$

$$\Rightarrow y(x, y) = [A\cos kx + B\sin kx]\,[C\cos(ckt) + D\sin(ckt)]$$

This is one solution for a particular value of 'k', but different values of 'k' will generate different solutions. The general solution is obtained by superposing them as :

$$\mathbf{y(x, t) = \Sigma_k\, [A_k \cos kx + B_k \sin kx]\, [C_k \cos(ckt) + D_k \sin(ckt)]}$$

Note that the constants 'A_k', 'B_k', etc., may differ for different values of 'k'.

At x = 0, y = 0 for all values of t $\Rightarrow$ $0 = \Sigma_k A_k [C_k \cos(ckt) + D_k \sin(ckt)]$

$$\Rightarrow\ A_k = 0$$

$$\Rightarrow\ y(x, t) = \Sigma_k B_k \sin kx\, [C_k \cos(ckt) + D_k \sin(ckt)]$$

At x = L, y = 0 for all values of y $\Rightarrow$ either $B_k = 0$, or, sin kL = 0, but both A_k and $B_k = 0$ will lead to the 'trivial solution' $y(x, t) = 0$ for all x and t, which means that the wire is not vibrating at all.

So, we turn towards the other choice : $\sin ka = 0 \Rightarrow kL = n\pi$, or, $\mathbf{k = n\pi/L}$.

We see, how the boundary condition can restrict the possible choices for 'k'.

Now, $y(x, t) = \Sigma_n B_n \sin(n\pi x/L) [C_n \cos(n\pi ct/L) + D_n \sin(n\pi ct/L)]$.

We have replaced 'k' by($n\pi/L$) and re-parametrized the constants 'A_k', 'B_k', etc., as 'A_n', 'B_n', etc.

We may absorb the const. B_n in C_n and D_n , calling : $B_n C_n = C_n'$ and $B_n D_n = D_n'$, so that :

$$\mathbf{y(x, t) = \Sigma_n \sin(n\pi x/L) [C_n' \cos(n\pi ct/L) + D_n' \sin(n\pi ct/L)].}$$

This is the general solution (standing wave) for vibration of a stretched string, fixed at both ends. **Struck String :**

Now suppose, we have an **initial condition** : (iii) $y(x, t) = 0$ at $t = 0$ for all values of x.

This will imply : $0 = \Sigma_n \sin(n\pi x/L) C_n' = 0 \Rightarrow C_n' = 0$

$$\mathbf{\Rightarrow y(x, t) = \Sigma_n D_n' \sin(n\pi x/L) \sin(n\pi ct/L)].}$$

Plucked String :

If instead, we have the **initial condition** : (iii) $\partial y/\partial t = 0$ at $t = 0$ for all values of x,

$\partial y/\partial t = \Sigma_n \sin(n\pi x/L) \times (n\pi c/L) [- C_n' \sin(n\pi ct/L) + D_n' \cos(n\pi ct/L)]$

$\Rightarrow 0 = \Sigma_n \sin(n\pi x/L) \times (n\pi c/L) D_n'$

$\Rightarrow D_n' = 0$

$$\mathbf{\Rightarrow y(x, t) = \Sigma_n C_n' \sin(n\pi x/L) \cos(n\pi ct/L)].}$$

To evaluate the remaining constants, we shall require another set of initial conditions.

Suppose, in case of a plucked string, the initial shape of the wire is given as F(x).

At $t = 0$, $y(x, t) = \Sigma_n C_n' \sin(n\pi x/L) = F(x)$

$\Rightarrow$ F(x) is already expanded in a Fourier sin series

$\Rightarrow$ $\mathbf{C_n' = (2/L) \int F(x) \sin(n\pi x/L)\, dx}$, **between the limits 0 and L.**