Superposition of Simple harmonic motions:

(a) Motion along the same straight line (direction) of same frequency (time period) but different phases and amplitudes:

Suppose two vibrations are given by

$$x_1 = a_1 \cos(\omega t-\varepsilon_1)$$

$$x_2 = a_2 \cos(\omega t-\varepsilon_2)$$

The resultant vibration is given by

$$X = x_1+ x_2= a_1 \cos(\omega t-\varepsilon_1) + a_2 \cos(\omega t-\varepsilon_2)$$

$$= a_1 \cos \omega t \cos \varepsilon_1 + a_1 \sin \omega t \sin \varepsilon_1 + a_2 \cos \omega t \cos \varepsilon_2 + a_2 \sin \omega t \sin \varepsilon_2$$

$$= (a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2) \cos \omega t + (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2) \sin \omega t$$

Let $a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2 = A \cos \delta$ and $a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2 = A \sin \delta$, where A and δ are two constants given by

$$A^2 = (a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2)^2 + (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2)^2$$

$$= a_1^2 + a_2^2 + 2 a_1 a_2 \cos (\varepsilon_1 - \varepsilon_2)$$

and
$$\tan\delta = \frac{a_1 Sin\varepsilon_1 + a_2 Sin\varepsilon_2}{a_1 Cos\varepsilon_1 + a_2 Cos\varepsilon_2}$$

Therefore, $x = A \cos (\omega t- \delta)$

Thus the resultant vibration has the same frequency (time period) as that of the component vibrations.

If instead of two vibrations there are several vibrations of different amplitudes and phases but of same frequency (time period), the resultant vibration can be similarly deduced.

$$X = a_1 \cos(\omega t-\varepsilon_1) + a_2 \cos(\omega t-\varepsilon_2)+a_3 \cos(\omega t-\varepsilon_3) + ...$$

$$= (a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2 + a_3 \cos \varepsilon_3 + ...) \cos \omega t + (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2 + a_3 \sin \varepsilon_3 + ...) \sin \omega t$$

Putting $a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2 + a_3 \cos \varepsilon_3 + ... = A \cos \delta$ and $a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2 + a_3 \sin \varepsilon_3 + ... = A \sin \delta$ where $A^2 = (a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2 + a_3 \cos \varepsilon_3 + ...)^2 + (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2 + a_3 \sin \varepsilon_3 + ...)^2$

and $$\tan\delta = \frac{a_1 Sin\varepsilon_1 + a_2 Sin\varepsilon_2 + a_3 Sin\varepsilon_3 + ...}{a_1 Cos\varepsilon_1 + a_2 Cos\varepsilon_2 + a_3 Cos\varepsilon_3 \ + ...}$$

Therefore, $x= A \cos(\omega t- \delta))$

(b) Two vibrations of slightly different frequencies (time period) along same straight line: Beats

Let $x_1=a_1 \cos(\omega_1 t-\varepsilon_1)$ and $x_2 = a_2 \cos\{(\omega_1+\omega_2)t-\varepsilon_2)\}$, where ω_2 is small number.

Frequencies of the two vibrations are given by $N_1= \omega_1/2\pi$ and $N_2=(\omega_1+\omega_2)/ 2\pi$.

Let us put $- \omega_2 t+\varepsilon_2 = \varepsilon_2'$

Therefore, $x_1=a_1 \cos(\omega_1 t-\varepsilon_1)$ and $x_2 = a_2 \cos(\omega_1 t- \varepsilon_2')$

Therefore, $x = x_1+ x_2 = A \cos(\omega_1 t- \delta)$, where

$$A^2 = a_1^2 + a_2^2 + 2 a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2')$$

and $\tan\delta = \dfrac{a_1 Sin\varepsilon_1 + a_2 Sin\varepsilon_2'}{a_1 Cos\varepsilon_1 + a_2 Cos\varepsilon_2'}$

Thus when $\varepsilon_1 - \varepsilon_2' = \varepsilon_1 - \varepsilon_2+ \omega_2 t = (2s+1)\pi$, where s = 0,1,2,3,...........etc.

$$A = a_1-a_2$$

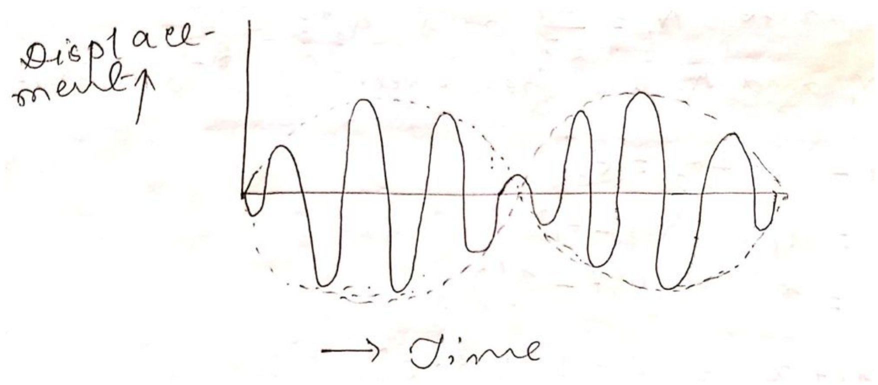
and when $\varepsilon_1 - \varepsilon_2' = \varepsilon_1 - \varepsilon_2+ \omega_2 t = 2s\pi$

$$A = a_1+a_2$$

Hence the amplitude of the resultant vibration changes between the limits (a_1-a_2) and (a_1+a_2).

If $a_1 = a_2 = a$ then the limits will be 0 and 2a. Thus the amplitude of the resultant vibration changes periodically with frequency equal to $\omega_2/ 2\pi$, which is the difference of the two component frequencies.

This phenomenon is called beats. It is observed when tuning fork or any two sources of sound of nearly equal frequencies are sounded together. The method of beats is a very important one in measurement of an unknown frequency.



The resultant vibration is not simple harmonic though the expression looks like the one. This is because neither the amplitude 'A' nor 'δ' are constant, both changes with time. Apperently, the vibrations seem to have frequency 'ω_1', with amplitude changing with a frequency 'ω_2'. As the time goes on the amplitude of the resultant vibration passes alternatively through the maximum value (a_1+a_2) and the minimum value (a_1-a_2). The time between two successive maxima (or minima) of resultant amplitude is) $2\pi/ \omega_2 = 1/n_2$. To observe the nature of motion clearly 'n_2' should be small.

Note: In the case of sound waves, the intensity depends on the square of amplitude. Hence when such superposition occurs, the intensity passes through maxima and minima with a frequency given by the difference of two superimposed frequencies. This phenomenon is called beats.

(c) Superposition of two vibrations at right angles to each other, time periods (frequencys) equal:

Let the two vibrations be

$$x = a_1 \text{Cos}\,(\omega t-\varepsilon_1) \text{ and } y = a_2 \text{Cos}\,(\omega t-\varepsilon_2)$$

$y = a_2 \text{Cos}\,\{(\omega t-\varepsilon_1)+(\varepsilon_1 - \varepsilon_2)\}$

$= a_2 \text{Cos}\,(\omega t-\varepsilon_1)\,\text{Cos}\,(\varepsilon_1 - \varepsilon_2) - a_2 \text{Sin}\,(\omega t-\varepsilon_1)\,\text{Sin}\,(\varepsilon_1 - \varepsilon_2)$

$= a_2 . \dfrac{x}{a_1} \text{Cos}\,(\varepsilon_1 - \varepsilon_2) - a_2 . \sqrt{1-\dfrac{x^2}{a_1^2}}\,\text{Sin}\,(\varepsilon_1 - \varepsilon_2)$

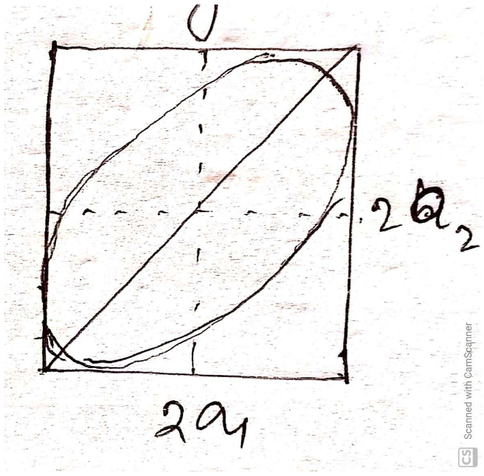
Therefore, $\sqrt{a_1^2 - x^2}\,Sin(\varepsilon_1 - \varepsilon_2) = \text{x}.\text{Cos}(\varepsilon_1 - \varepsilon_2) - \text{y}.\dfrac{a_1}{a_2}$

Or, $(a_1^2 - x^2)Sin^2(\varepsilon_1 - \varepsilon_2) = \text{x}^2\text{Cos}^2(\varepsilon_1 - \varepsilon_2) - 2\,\text{xy}.\dfrac{a_1}{a_2}Cos(\varepsilon_1 - \varepsilon_2) + \text{y}^2.\dfrac{a_1^2}{a_2^2}$

Or, $a_1^2 Sin^2(\varepsilon_1 - \varepsilon_2) = \text{x}^2 - 2\,\text{xy}.\dfrac{a_1}{a_2}Cos(\varepsilon_1 - \varepsilon_2) + \text{y}^2.\dfrac{a_1^2}{a_2^2}$

Or, $Sin^2(\varepsilon_1 - \varepsilon_2) = \dfrac{\text{x}^2}{a_1^2} + \dfrac{\text{y}^2}{a_2^2} - \dfrac{2\,\text{xy}}{a_1 a_2}Cos(\varepsilon_1 - \varepsilon_2)$

This is the resultant motion. This equation represents an ellipse. Thus the motion in general is an ellipse and position of the particle at any instant depends on a_1, a_2, ε_1 and ε_2.



Special cases:

1. Let $\varepsilon_1 - \varepsilon_2 = 0$

Therefore, $\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} = 0$

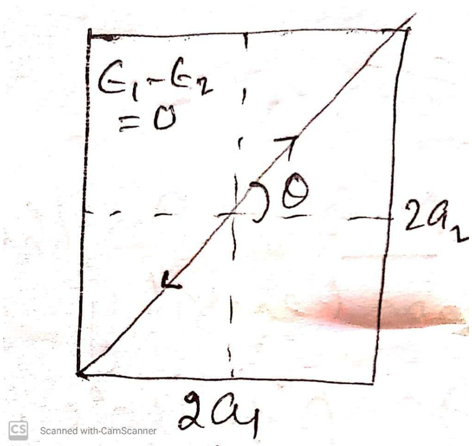
Or, $(\frac{x}{a_1} - \frac{y}{a_2})^2 = 0$

Or, $\frac{x}{a_1} - \frac{y}{a_2} = 0$

Or, $y = \frac{a_2}{a_1} x$

This represents a straight line passing through the origin making an angle θ with the X-axis such that

$\tan\theta = \frac{a_2}{a_1}$



2. If $\varepsilon_1 - \varepsilon_2 = \pi$

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{2xy}{a_1 a_2} = 0$$

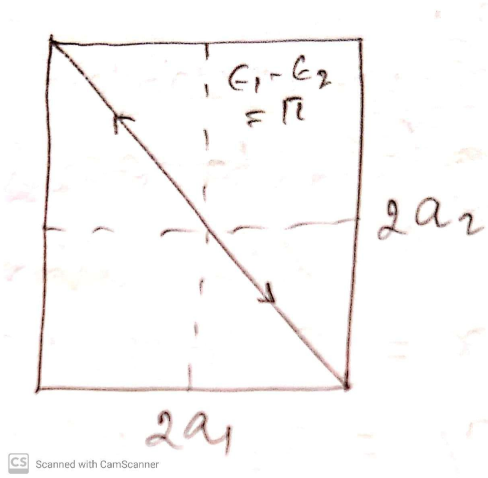
Or, $(\frac{x}{a_1} + \frac{y}{a_2})^2 = 0$

Or, $\frac{x}{a_1} + \frac{y}{a_2} = 0$

Or, $y = -\frac{a_2}{a_1} x$

This represents a straight line passing through the origin making an angle θ with the X-axis such that

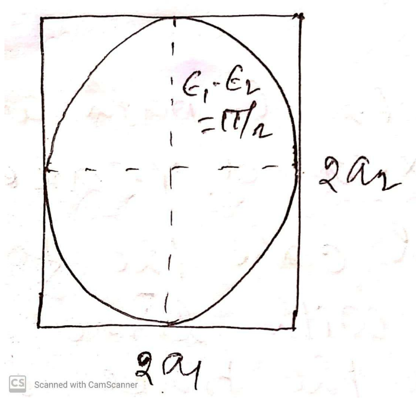
$$\tan\theta = -\frac{a_2}{a_1}$$



3. If $\varepsilon_1 - \varepsilon_2 = \pi/2$

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$$

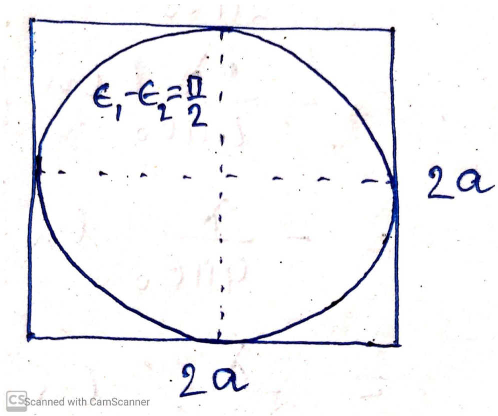
This represents an ellipse with two axes coinciding with X and Y-axes.



4. If $a_1 = a_2 = a$ and $\varepsilon_1 - \varepsilon_2 = \pi/2$

$$x^2 + y^2 = a^2$$

It represents a circle of radius 'a'. Motion in this case is a uniform circular motion with angular velocity 'ω'.



(d) Two vibrations of slightly different frequencies at right angles to each other

Let the two vibrations be

$x = a_1 \, Cos\,(\omega_1 t - \varepsilon_1)$ and $y = a_2 \, Cos\,\{(\omega_1 + \omega_2)t - \varepsilon_2)\}$, where ω_2 is a small number.

Let us write $-\omega_2 t + \varepsilon_2 = \varepsilon_2'$

Therefore, $y = a_2 \, Cos\,(\omega_1 t - \varepsilon_2')$

$$\therefore \; Sin^2(\varepsilon_1 - \varepsilon_2') = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} Cos(\varepsilon_1 - \varepsilon_2')$$

$$\therefore \; Sin^2(\omega_2 t + \varepsilon_1 - \varepsilon_2) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} Cos(\omega_2 t + \varepsilon_1 - \varepsilon_2)$$

Shape of the resultant path will depend on the value of (ω_2 t+ $\varepsilon_1 -\ \varepsilon_2$). If ω_2 t+ $\varepsilon_1 -\ \varepsilon_2$ = sπ, where s is an integer.

$$\therefore\ \frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}\mp\frac{2xy}{a_1a_2}=0$$

Or, $(\frac{x}{a_1}\mp\frac{y}{a_2})^2=0$

Or, $\frac{x}{a_1}\mp\frac{y}{a_2}=0$

$$\therefore\ y=\pm\frac{a_2}{a_1}x$$

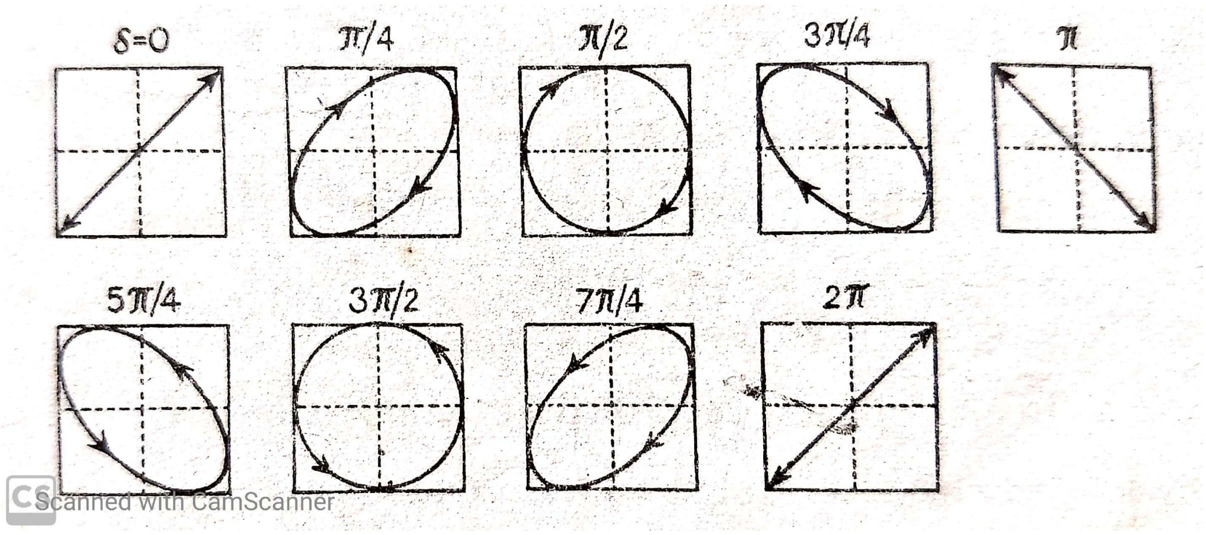
Hence the resultant path will be a straight line passing through the origin.

If ω_2 t+ $\varepsilon_1 -\ \varepsilon_2$ = (2s+1)π/2,

$$\therefore\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}=1$$

Hence the resultant path will be an ellipse with axes along X- and Y-axes.

In the general case the form of the curve will be elliptical. With time (ω_2 t+ $\varepsilon_1 -\ \varepsilon_2$) will change and hence the trace of the path described by the particle will gradually change its pattern and greater the difference between two time periods (or, smaller the difference between two component frequencies), the more quickly the nature of the curve will change.



At any instant ‘t’

$$Sin^2(\omega_2 t+\varepsilon_1 - \varepsilon_2) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} Cos(\omega_2 t+\varepsilon_1 - \varepsilon_2)$$

At any other later instant ‘t_1’

$$Sin^2(\omega_2 t_1+\varepsilon_1 - \varepsilon_2) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} Cos(\omega_2 t_1+\varepsilon_1 - \varepsilon_2)$$

The two curves will be identical if,

$$\omega_2 t_1+\varepsilon_1 - \varepsilon_2 = \omega_2 t+\varepsilon_1 - \varepsilon_2 + 2\pi$$

Or, $\omega_2(t_1 - t) = 2\pi$

$$\therefore\ t_1 - t = \frac{2\pi}{\omega_2}$$

This is the time difference between the formations of two identical curves.